Voigt transforms

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Voigt notation is a common method, particularly in solid mechanics, for expressing symmetric tensors as a lower order tensor. As a basic example, it produces the mapping

$$\begin{pmatrix} T^{11} & T^{12} & T^{13} \\ T^{12} & T^{22} & T^{23} \\ T^{13} & T^{23} & T^{33} \end{pmatrix} \mapsto \begin{pmatrix} T^{11} \\ T^{22} \\ T^{33} \\ T^{23} \\ T^{13} \\ T^{12} \end{pmatrix} .$$

For a simple look-up reference of the standard convention, see Sec. 4. However, for a deeper understanding, this tutorial details the mathematical foundation of Voigt notation to expose it subtleties [2]. In particular, it should be noted that even if the original space is described by an identity metric, the 'Voigt space' metric is not necessarily also the identity, and care needs to be taken when performing contractions or raising/lowering indices.

1 Tensors

First, let us quickly review tensors. Starting with a vector space V of dimension n over field $\mathbb{F}$ with basis $\{\vec{e}_i\}_{i=1}^n$, and its dual space V^* with basis $\{\vec{e}^{\,i}\}_{i=1}^n$, we define the tensor space $\mathcal{T}_q^p(V)$ of contravariant order p and covariant order q as the set of tensors (multilinear maps)

$$\mathbf{T} : \underbrace{V^* \times V^* \times \cdots}_{p} \times \underbrace{V \times V \times \cdots}_{q} \longrightarrow \mathbb{F} ,$$

$$\mathbf{T} = \sum_{i_1=1}^{n} \cdots \sum_{i_p=1}^{n} \sum_{j_1=1}^{n} \cdots \sum_{j_q=1}^{n} T^{i_1 \cdots i_p}_{j_1 \cdots j_q} \, \vec{e}_{i_1} \otimes \cdots \otimes \vec{e}_{i_n} \otimes \vec{e}^{\,j_1} \otimes \cdots \otimes \vec{e}^{\,j_n} .$$

We define the metric for V as

$$\mathbf{g} = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{i\,j} \, \vec{e}^{\,i} \otimes \vec{e}^{\,j} ,$$

so that $T^{\cdots}{}_{i}{}^{\cdots} = \sum_{j=1}^{n} g_{i\,j} \, T^{\cdots j \cdots}$.

2 Voigt transform

A Voigt transform on a pair of symmetric tensor indices reduces it to a single index,

$$\left(\mathbf{T} : \cdots \times V \times V \times \cdots \to \mathbb{F}\right) \mapsto \left(\tilde{\mathbf{T}} : \cdots \times W \times \cdots \to \mathbb{F}\right)$$
$$T_{\cdots\,(i\,j)\,\cdots} \mapsto \tilde{T}_{\cdots\,a\,\cdots}$$

where W, of dimension $m = \frac{n^2+n}{2}$, is a linear subspace of $V \otimes V$, has basis $\{\vec{f}_a\}_{a=1}^{m}$, and metric $\mathbf{h}$. Parentheses denote symmetrization $T_{(i\,j)} = \frac{1}{2}(T_{i\,j} + T_{j\,i})$, which is simply $T_{i\,j}$ if already symmetric. For a faithful Voigt representation, we require that the tensors provide an equivalent mapping,

$$\cdots \sum_{i=1}^{n} \sum_{j=1}^{n} \cdots \; T^{\cdots\,(i\,j)\,\cdots} \; \cdots \vec{e}_i \otimes \vec{e}_j \cdots = \mathbf{T} = \tilde{\mathbf{T}} = \cdots \sum_{a=1}^{m} \cdots \; \tilde{T}^{\cdots\,a\,\cdots} \; \cdots \vec{f}_a \cdots \, , \tag{1}$$

where expressions filling in the corresponding ellipses are unchanged.

In order to proceed we define a general relationship between basis vectors $\vec{f}_a$ and $\vec{e}_i$,

$$\vec{f}_a = \begin{cases} \vec{e}_a \otimes \vec{e}_a & : \quad a \in [1, n] \subset \mathbb{Z} \\ \alpha \left(\vec{e}_{i_a} \otimes \vec{e}_{j_a} + \vec{e}_{j_a} \otimes \vec{e}_{i_a}\right) & : \quad a \in [n+1, m] \subset \mathbb{Z} \end{cases} ,$$

where $i_a \neq j_a$ are determined by a Voigt ordering convention (see Sec. 2.1). Putting this into (1) we discern

$$\tilde{T}^{\cdots\,a\,\cdots} = \begin{cases} T^{\cdots\,a\,a\,\cdots} & : \quad a \in [1, n] \subset \mathbb{Z} \\ \frac{1}{\alpha} T^{\cdots\,i_a\,j_a\,\cdots} & : \quad a \in [n+1, m] \subset \mathbb{Z} \end{cases} .$$

We now wish to impose that contractions behave properly in Voigt notation, i.e.

$$\sum_{i=1}^{n} \sum_{j=1}^{n} T^{\cdots\,i\,j\,\cdots} U_{\cdots\,i\,j\,\cdots} = \sum_{a=1}^{m} \tilde{T}^{\cdots\,a\,\cdots} \tilde{U}_{\cdots\,a\,\cdots} \, .$$

Expanding this out we determine the rules for covariant tensor elements to be

$$\tilde{T}_{\cdots\,a\,\cdots} = \begin{cases} T_{\cdots\,a\,a\,\cdots} & : \quad a \in [1, n] \subset \mathbb{Z} \\ 2\,\alpha\, T_{\cdots\,i_a\,j_a\,\cdots} & : \quad a \in [n+1, m] \subset \mathbb{Z} \end{cases} .$$

The dual basis vectors are thus

$$\vec{f}^a = \begin{cases} \vec{e}^a \otimes \vec{e}_a & : \quad a \in [1, n] \subset \mathbb{Z} \\ \frac{1}{2\,\alpha} \left(\vec{e}^{i_a} \otimes \vec{e}^{j_a} + \vec{e}^{j_a} \otimes \vec{e}^{i_a}\right) & : \quad a \in [n+1, m] \subset \mathbb{Z} \end{cases} .$$

With these mappings we can also derive the metric $\mathbf{h}$ on W to be

$$h_{a\,b} = \begin{cases} g_{a\,b}\, g_{a\,b} & : \quad a \in [1, n] \wedge b \in [1, n] \\ 2\,\alpha\, g_{a\,i_b}\, g_{a\,j_b} & : \quad a \in [1, n] \wedge b \in [n+1, m] \\ 2\,\alpha\, g_{i_a\,b}\, g_{j_a\,b} & : \quad a \in [n+1, m] \wedge b \in [1, n] \\ 2\,\alpha^2 \left(g_{i_a\,i_b}\, g_{j_a\,j_b} + g_{i_a\,j_b}\, g_{j_a\,i_b}\right) & : \quad a \in [n+1, m] \wedge b \in [n+1, m] \end{cases} .$$

2.1 Ordering

The matching of tensor indices to their Voigt counterpart follows a spiral pattern, starting along the diagonal, then going around the upper right triangle of the symmetric tensor. An example for $n = 5$ is given in Fig. 1. This provides the mapping $a \leftrightarrow \{i_a, j_a\}$.

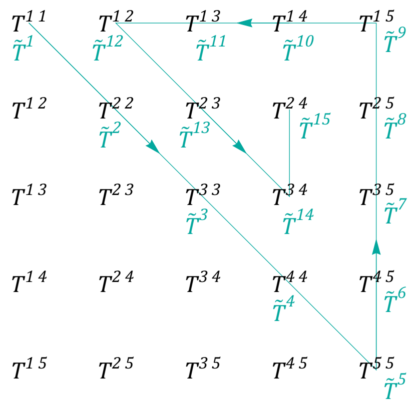

Figure 1: Voigt ordering for a 5×5 symmetric tensor.

2.2 Multiple pairs

Tensors with multiple pairs of symmetric indices can have each Voigt transformed. For example, the transformation rules for $T^{(i\,j)\,(k\,l)} \mapsto \tilde{T}^{a\,b}$ are

$$\tilde{T}^{a\,b} = \begin{cases} T^{a\,a\,b\,b} & : \quad a \in [1,n] \wedge b \in [1,n] \\ \frac{1}{\alpha}\, T^{a\,a\,i_b\,j_b} & : \quad a \in [1,n] \wedge b \in [n+1,m] \\ \frac{1}{\alpha}\, T^{i_a\,j_a\,b\,b} & : \quad a \in [n+1,m] \wedge b \in [1,n] \\ \frac{1}{\alpha^2}\, T^{i_a\,j_a\,i_b\,j_b} & : \quad a \in [n+1,m] \wedge b \in [n+1,m] \end{cases} ,$$

$$\tilde{T}_{a\,b} = \begin{cases} T_{a\,a\,b\,b} & : \quad a \in [1,n] \wedge b \in [1,n] \\ 2\,\alpha\, T_{a\,a\,i_b\,j_b} & : \quad a \in [1,n] \wedge b \in [n+1,m] \\ 2\,\alpha\, T_{i_a\,j_a\,b\,b} & : \quad a \in [n+1,m] \wedge b \in [1,n] \\ 4\,\alpha^2\, T_{i_a\,j_a\,i_b\,j_b} & : \quad a \in [n+1,m] \wedge b \in [n+1,m] \end{cases} .$$

A complete Voigt transform on a tensor, with all pairs of indices symmetric, maps $\mathcal{T}^p(V) \rightarrow \mathcal{T}^{p/2}(W)$.

2.3 Multi-space tensor representation

Voigt transforms can be represented by a multi-space tensor that acts of different vector spaces. In particular, the contravariant Voigt transform is performed by

$$\mathbf{\Xi} : W^* \times V \times V \longrightarrow \mathbb{F} \, ,$$

such that

$$\tilde{T}^a = \sum_{i=1}^{n} \sum_{j=1}^{n} \Xi^a_{\ i\,j}\, T^{i\,j} \ , \qquad\qquad \Xi^a_{\ i\,j} = \Xi^a_{\ j\,i} \ .$$

The vector space V has dimension n and metric $\mathbf{g}$, whilst the vector space W has dimension $m = \frac{n^2+n}{2}$ and metric $\mathbf{h}$. Thus, the covariant Voigt transform is

$$\tilde{T}_a = \sum_{i=1}^{n} \sum_{j=1}^{n} \Xi_a^{\ i\,j}\, T_{i\,j} \ , \qquad\qquad \Xi_a^{\ i\,j} = \sum_{b=1}^{m} \sum_{k=1}^{n} \sum_{l=1}^{n} h_{a\,b}\, g^{i\,k}\, g^{j\,l}\, \Xi^b_{\ k\,l} \ .$$

Using our derivation above

$$\Xi^a_{\ i\,j} = \begin{cases} 1 & : \quad a \in [1,n] \wedge a = i = j \\ \frac{1}{2\,\alpha} & : \quad a \in [n+1,m] \wedge \{i_a, j_a\} = \{i, j\} \\ 0 & : \quad \text{otherwise} \end{cases} \ ,$$

$$\Xi_a^{\ i\,j} = \begin{cases} 1 & : \quad a \in [1,n] \wedge a = i = j \\ \alpha & : \quad a \in [n+1,m] \wedge \{i_a, j_a\} = \{i, j\} \\ 0 & : \quad \text{otherwise} \end{cases} \ ,$$

where $\{i_a, j_a\}$ is the Voigt-ordered pair of indices for a and sets are unsorted. The factor of a half comes from the double-counting of off-diagonal symmetric elements.

By the nature of its construction, $\boldsymbol{\Xi}$ is self-inverse,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \Xi^a_{\ i\,j}\, \Xi_b^{\ i\,j} = \delta^a_b \ , \qquad\qquad \sum_{a=1}^{m} \Xi^a_{\ i\,j}\, \Xi_a^{\ k\,l} = \frac{1}{2} \left(\delta^k_i\, \delta^l_j + \delta^l_i\, \delta^k_j \right) \ ,$$

and the inverse Voigt transform is given by the same tensor,

$$T^{i\,j} = \sum_{a=1}^{m} \Xi_a^{\ i\,j}\, \tilde{T}^a \ , \qquad\qquad T_{i\,j} = \sum_{a=1}^{m} \Xi^a_{\ i\,j}\, \tilde{T}_a \ .$$

3 Three dimensions

In the case of $n = 3$ and using standard Voigt ordering $(2\,3, 1\,3, 1\,2)$,

$$i_4\, j_4 = 2\,3 \ , \qquad\qquad i_5\, j_5 = 1\,3 \ , \qquad\qquad i_6\, j_6 = 1\,2 \ .$$

Our basis vectors are

$$\begin{aligned} \vec{f}_1 &= \vec{e}_1 \otimes \vec{e}_1 \ , & \vec{f}_4 &= \alpha \left(\vec{e}_2 \otimes \vec{e}_3 + \vec{e}_3 \otimes \vec{e}_2 \right) \ , \\ \vec{f}_2 &= \vec{e}_2 \otimes \vec{e}_2 \ , & \vec{f}_5 &= \alpha \left(\vec{e}_1 \otimes \vec{e}_3 + \vec{e}_3 \otimes \vec{e}_1 \right) \ , \\ \vec{f}_3 &= \vec{e}_3 \otimes \vec{e}_3 \ , & \vec{f}_6 &= \alpha \left(\vec{e}_1 \otimes \vec{e}_2 + \vec{e}_2 \otimes \vec{e}_1 \right) \ , \end{aligned}$$

$$\begin{aligned} \vec{f}^1 &= \vec{e}^1 \otimes \vec{e}^1 \ , & \vec{f}^4 &= \tfrac{1}{2\,\alpha} \left(\vec{e}^2 \otimes \vec{e}^3 + \vec{e}^3 \otimes \vec{e}^2 \right) \ , \\ \vec{f}^2 &= \vec{e}^2 \otimes \vec{e}^2 \ , & \vec{f}^5 &= \tfrac{1}{2\,\alpha} \left(\vec{e}^1 \otimes \vec{e}^3 + \vec{e}^3 \otimes \vec{e}^1 \right) \ , \\ \vec{f}^3 &= \vec{e}^3 \otimes \vec{e}^3 \ , & \vec{f}^6 &= \tfrac{1}{2\,\alpha} \left(\vec{e}^1 \otimes \vec{e}^2 + \vec{e}^2 \otimes \vec{e}^1 \right) \ . \end{aligned}$$

The tensor coefficient transformation rules are

$$\begin{aligned} \tilde{T}^1 &= T^{11}\,, & \tilde{T}^4 &= \tfrac{1}{\alpha}\,T^{23}\,,\\ \tilde{T}^2 &= T^{22}\,, & \tilde{T}^5 &= \tfrac{1}{\alpha}\,T^{13}\,,\\ \tilde{T}^3 &= T^{33}\,, & \tilde{T}^6 &= \tfrac{1}{\alpha}\,T^{12}\,, \end{aligned}$$

$$\begin{aligned} \tilde{T}_1 &= T_{11}\,, & \tilde{T}_4 &= 2\,\alpha\,T_{23}\,,\\ \tilde{T}_2 &= T_{22}\,, & \tilde{T}_5 &= 2\,\alpha\,T_{13}\,,\\ \tilde{T}_3 &= T_{33}\,, & \tilde{T}_6 &= 2\,\alpha\,T_{12}\,, \end{aligned}$$

and the Voigt transform multi-spaced tensor is

$$(\Xi^a{}_{i\,j}) = \left(\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&\frac{1}{2\,\alpha}\\0&\frac{1}{2\,\alpha}&0\end{pmatrix}\begin{pmatrix}0&0&\frac{1}{2\,\alpha}\\0&0&0\\\frac{1}{2\,\alpha}&0&0\end{pmatrix}\begin{pmatrix}0&\frac{1}{2\,\alpha}&0\\\frac{1}{2\,\alpha}&0&0\\0&0&0\end{pmatrix}\right),$$

with the major column expressing the a index.

The metric on W is

$$(h_{a\,b}) = \begin{pmatrix} g_{11}\,g_{11} & g_{12}\,g_{12} & g_{13}\,g_{13} & 2\,\alpha\,g_{12}\,g_{13} & 2\,\alpha\,g_{11}\,g_{13} & 2\,\alpha\,g_{11}\,g_{12} \\ g_{12}\,g_{12} & g_{22}\,g_{22} & g_{23}\,g_{23} & 2\,\alpha\,g_{22}\,g_{23} & 2\,\alpha\,g_{12}\,g_{23} & 2\,\alpha\,g_{12}\,g_{22} \\ g_{13}\,g_{13} & g_{23}\,g_{23} & g_{33}\,g_{33} & 2\,\alpha\,g_{23}\,g_{33} & 2\,\alpha\,g_{13}\,g_{33} & 2\,\alpha\,g_{13}\,g_{23} \\ 2\,\alpha\,g_{12}\,g_{13} & 2\,\alpha\,g_{22}\,g_{23} & 2\,\alpha\,g_{33}\,g_{33} & 2\,\alpha^2\,({g_{23}}^2 + g_{22}\,g_{33}) & 2\,\alpha^2\,(g_{13}\,g_{23} + g_{12}\,g_{33}) & 2\,\alpha^2\,(g_{13}\,g_{22} + g_{12}\,g_{23}) \\ 2\,\alpha\,g_{11}\,g_{13} & 2\,\alpha\,g_{12}\,g_{23} & 2\,\alpha\,g_{13}\,g_{33} & 2\,\alpha^2\,(g_{13}\,g_{23} + g_{12}\,g_{33}) & 2\,\alpha^2\,({g_{13}}^2 + g_{11}\,g_{33}) & 2\,\alpha^2\,(g_{12}\,g_{13} + g_{11}\,g_{23}) \\ 2\,\alpha\,g_{11}\,g_{12} & 2\,\alpha\,g_{12}\,g_{22} & 2\,\alpha\,g_{13}\,g_{23} & 2\,\alpha^2\,(g_{13}\,g_{22} + g_{12}\,g_{23}) & 2\,\alpha^2\,(g_{12}\,g_{13} + g_{11}\,g_{23}) & 2\,\alpha^2\,({g_{12}}^2 + g_{11}\,g_{22}) \end{pmatrix}.$$

For the common case of $V = \mathbb{R}^3$ and $(g_{i\,j}) = \mathbb{I}_3$, we get

$$(h_{a\,b}) = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&2\,\alpha^2&0&0\\ 0&0&0&0&2\,\alpha^2&0\\ 0&0&0&0&0&2\,\alpha^2 \end{pmatrix},$$

which in general is not equal to $\mathbb{I}_6$. Thus whilst indices on T can be raised or lowered 'freely,' the same is not true for $\tilde{T}$ and care must be taken to ensure only covariant and contravariant indices are contracted together.

4 Solid mechanics convention

In solid mechanics it is typical to take $\alpha = 1$, denote the contravariant Voigt transform as ‘stress-like’, and the covariant Voigt transform as ‘strain-like’. In this setting many authors do not properly keep track of raised/lowered indices and instead ensure that any contractions in Voigt notation take place between a stress-like transformed tensor and a strain-like transformed tensor. As such, stress $\sigma_{(ij)}$, stiffness $c_{(ij)(kl)}$, and piezoelectric coupling $e_{i(jk)}$ transform stress-like; whilst strain $\epsilon_{(ij)}$, compliance $s_{(ij)(kl)}$, and piezoelectric coupling $d_{i(jk)}$ transform strain-like.

For reference,

$$\begin{pmatrix} \tilde{\sigma}^1 \\ \tilde{\sigma}^2 \\ \tilde{\sigma}^3 \\ \tilde{\sigma}^4 \\ \tilde{\sigma}^5 \\ \tilde{\sigma}^6 \end{pmatrix} \underset{\substack{\text{stress}\\\text{like}}}{=} \begin{pmatrix} \sigma^{11} \\ \sigma^{22} \\ \sigma^{33} \\ \sigma^{23} \\ \sigma^{13} \\ \sigma^{12} \end{pmatrix}, \qquad \begin{pmatrix} \tilde{\epsilon}_1 \\ \tilde{\epsilon}_2 \\ \tilde{\epsilon}_3 \\ \tilde{\epsilon}_4 \\ \tilde{\epsilon}_5 \\ \tilde{\epsilon}_6 \end{pmatrix} \underset{\substack{\text{strain}\\\text{like}}}{=} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\,\epsilon_{23} \\ 2\,\epsilon_{13} \\ 2\,\epsilon_{12} \end{pmatrix},$$

$$\begin{pmatrix} \tilde{c}^{11} & \tilde{c}^{12} & \tilde{c}^{13} & \tilde{c}^{14} & \tilde{c}^{15} & \tilde{c}^{16} \\ \tilde{c}^{21} & \tilde{c}^{22} & \tilde{c}^{23} & \tilde{c}^{24} & \tilde{c}^{25} & \tilde{c}^{26} \\ \tilde{c}^{31} & \tilde{c}^{32} & \tilde{c}^{33} & \tilde{c}^{34} & \tilde{c}^{35} & \tilde{c}^{36} \\ \tilde{c}^{41} & \tilde{c}^{42} & \tilde{c}^{43} & \tilde{c}^{44} & \tilde{c}^{45} & \tilde{c}^{46} \\ \tilde{c}^{51} & \tilde{c}^{52} & \tilde{c}^{53} & \tilde{c}^{54} & \tilde{c}^{55} & \tilde{c}^{56} \\ \tilde{c}^{61} & \tilde{c}^{62} & \tilde{c}^{63} & \tilde{c}^{64} & \tilde{c}^{65} & \tilde{c}^{66} \end{pmatrix} \underset{\substack{\text{stress}\\\text{like}}}{=} \begin{pmatrix} c^{1111} & c^{1122} & c^{1133} & c^{1123} & c^{1113} & c^{1112} \\ c^{2211} & c^{2222} & c^{2233} & c^{2223} & c^{2213} & c^{2212} \\ c^{3311} & c^{3322} & c^{3333} & c^{3323} & c^{3313} & c^{3312} \\ c^{2311} & c^{2322} & c^{2333} & c^{2323} & c^{2313} & c^{2312} \\ c^{1311} & c^{1322} & c^{1333} & c^{1323} & c^{1313} & c^{1312} \\ c^{1211} & c^{1222} & c^{1233} & c^{1223} & c^{1213} & c^{1212} \end{pmatrix},$$

$$\begin{pmatrix} \tilde{s}_{11} & \tilde{s}_{12} & \tilde{s}_{13} & \tilde{s}_{14} & \tilde{s}_{15} & \tilde{s}_{16} \\ \tilde{s}_{21} & \tilde{s}_{22} & \tilde{s}_{23} & \tilde{s}_{24} & \tilde{s}_{25} & \tilde{s}_{26} \\ \tilde{s}_{31} & \tilde{s}_{32} & \tilde{s}_{33} & \tilde{s}_{34} & \tilde{s}_{35} & \tilde{s}_{36} \\ \tilde{s}_{41} & \tilde{s}_{42} & \tilde{s}_{43} & \tilde{s}_{44} & \tilde{s}_{45} & \tilde{s}_{46} \\ \tilde{s}_{51} & \tilde{s}_{52} & \tilde{s}_{53} & \tilde{s}_{54} & \tilde{s}_{55} & \tilde{s}_{56} \\ \tilde{s}_{61} & \tilde{s}_{62} & \tilde{s}_{63} & \tilde{s}_{64} & \tilde{s}_{65} & \tilde{s}_{66} \end{pmatrix} \underset{\substack{\text{strain}\\\text{like}}}{=} \begin{pmatrix} s_{1111} & s_{1122} & s_{1133} & 2\,s_{1123} & 2\,s_{1113} & 2\,s_{1112} \\ s_{2211} & s_{2222} & s_{2233} & 2\,s_{2223} & 2\,s_{2213} & 2\,s_{2212} \\ s_{3311} & s_{3322} & s_{3333} & 2\,s_{3323} & 2\,s_{3313} & 2\,s_{3312} \\ 2\,s_{2311} & 2\,s_{2322} & 2\,s_{2333} & 4\,s_{2323} & 4\,s_{2313} & 4\,s_{2312} \\ 2\,s_{1311} & 2\,s_{1322} & 2\,s_{1333} & 4\,s_{1323} & 4\,s_{1313} & 4\,s_{1312} \\ 2\,s_{1211} & 2\,s_{1222} & 2\,s_{1233} & 4\,s_{1223} & 4\,s_{1213} & 4\,s_{1212} \end{pmatrix},$$

$$\begin{pmatrix} \tilde{e}^{11} & \tilde{e}^{12} & \tilde{e}^{13} & \tilde{e}^{14} & \tilde{e}^{15} & \tilde{e}^{16} \\ \tilde{e}^{21} & \tilde{e}^{22} & \tilde{e}^{23} & \tilde{e}^{24} & \tilde{e}^{25} & \tilde{e}^{26} \\ \tilde{e}^{31} & \tilde{e}^{32} & \tilde{e}^{33} & \tilde{e}^{34} & \tilde{e}^{35} & \tilde{e}^{36} \end{pmatrix} \underset{\substack{\text{stress}\\\text{like}}}{=} \begin{pmatrix} e^{111} & e^{122} & e^{133} & e^{123} & e^{113} & e^{112} \\ e^{211} & e^{222} & e^{233} & e^{223} & e^{213} & e^{212} \\ e^{311} & e^{322} & e^{333} & e^{323} & e^{313} & e^{312} \end{pmatrix},$$

$$\begin{pmatrix} \tilde{d}_{11} & \tilde{d}_{12} & \tilde{d}_{13} & \tilde{d}_{14} & \tilde{d}_{15} & \tilde{d}_{16} \\ \tilde{d}_{21} & \tilde{d}_{22} & \tilde{d}_{23} & \tilde{d}_{24} & \tilde{d}_{25} & \tilde{d}_{26} \\ \tilde{d}_{31} & \tilde{d}_{32} & \tilde{d}_{33} & \tilde{d}_{34} & \tilde{d}_{35} & \tilde{d}_{36} \end{pmatrix} \underset{\substack{\text{strain}\\\text{like}}}{=} \begin{pmatrix} d_{111} & d_{122} & d_{133} & 2\,d_{123} & 2\,d_{113} & 2\,d_{112} \\ d_{211} & d_{222} & d_{233} & 2\,d_{223} & 2\,d_{213} & 2\,d_{212} \\ d_{311} & d_{322} & d_{333} & 2\,d_{323} & 2\,d_{313} & 2\,d_{312} \end{pmatrix},$$

5 Mandel notation

In Mandel notation, or ortho-normal representation, $\alpha = 1/\sqrt{2}$ and $(h_{ij}) = \mathbb{I}_m$ with covariant and contravariant transforms identical. This might have been a nicer convention.

6 Penrose diagrams

Voigt transforms can be depicted in Penrose tensor diagrams with the multi-spaced tensor Ξ at the junctions, as given in Fig. 2.

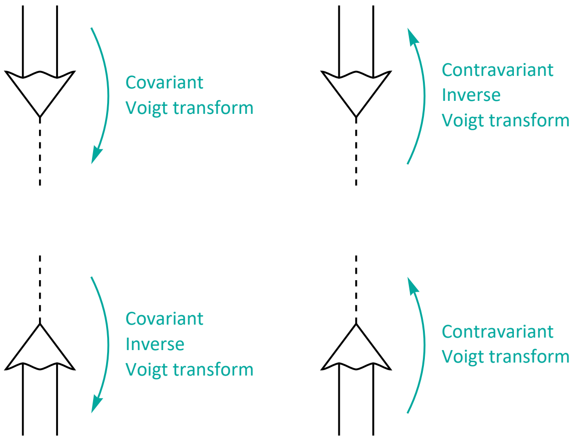

Figure 2: Voigt transforms in Penrose tensor diagrams. Solid lines represent indices of vector space V, whilst dashed lines represent indices of vector space W. The wiggly edge alludes to the index pair being symmetric.

References

[1] R. G. Povey, *Two-Dimensional Optomechanical Resonators in Gallium Arsenide*. PhD thesis, The University of Chicago, 2023. https://knowledge.uchicago.edu/record/7543.

[2] P. Helnwein, "Some remarks on the compressed matrix representation of symmetric second-order and fourth-order tensors," *Computer Methods in Applied Mechanics and Engineering* **190** no. 22-23, (Feb, 2001) 2753–2770. https://www.sciencedirect.com/science/article/pii/S0045782500002632.