Quantum simple harmonic oscillator

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The lowest-order non-trivial approximation to most anything is a harmonic oscillator.

1 Definitions

The Hamiltonian for a quantum simple harmonic oscillator is

$$\hat{H} = \hbar\,\Omega\left(\hat{a}^\dagger\,\hat{a} + \frac{1}{2}\right) = \frac{\hat{p}^2}{2\,m} + \frac{m\,\Omega^2\,\hat{x}^2}{2}\,, \tag{1}$$

with

$$\underbrace{m}_{\text{mass}}\,, \qquad \underbrace{\Omega}_{\text{angular resonance frequency}}\,,$$

$$\underbrace{\hat{x}}_{\text{position}} = \sqrt{\frac{\hbar}{2\,m\,\Omega}}\left(\hat{a}^\dagger + \hat{a}\right)\,, \qquad \underbrace{\hat{a}}_{\text{annihilation}} = \sqrt{\frac{m\,\Omega}{2\,\hbar}}\left(\hat{x} + \frac{\mathrm{i}}{m\,\Omega}\hat{p}\right)\,,$$

$$\underbrace{\hat{p}}_{\text{momentum}} = \mathrm{i}\sqrt{\frac{\hbar\,m\,\Omega}{2}}\left(\hat{a}^\dagger - \hat{a}\right)\,, \qquad \underbrace{\hat{a}^\dagger}_{\text{creation}} = \sqrt{\frac{m\,\Omega}{2\,\hbar}}\left(\hat{x} - \frac{\mathrm{i}}{m\,\Omega}\hat{p}\right)\,.$$

As observables, the position and momentum operators are Hermitian, $\hat{x}^\dagger = \hat{x}$ and $\hat{p}^\dagger = \hat{p}$.

In opposing bases we have

$$\hat{x}^\dagger = \hat{x} = \mathrm{i}\,\hbar\,\frac{\partial}{\partial p}\,, \qquad \hat{p}^\dagger = \hat{p} = -\mathrm{i}\,\hbar\,\frac{\partial}{\partial x}\,,$$

noting that within an inner product, see (39),

$$\left(\frac{\partial}{\partial x}\right)^\dagger = -\frac{\partial}{\partial x}\,, \qquad \left(\frac{\partial}{\partial p}\right)^\dagger = -\frac{\partial}{\partial p}\,.$$

2 Commutator relations

Some basic commutator identities are

$$[\hat{x}, \hat{p}] = \mathrm{i}\,\hbar\,, \qquad \left[\hat{a}, \hat{a}^\dagger\right] = 1\,.$$

$$\left[\hat{a}^\dagger\,\hat{a}, \hat{a}^\dagger\right] = \hat{a}^\dagger\,, \qquad \left[\hat{a}^\dagger\,\hat{a}, \hat{a}\right] = -\,\hat{a}\,. \tag{2}$$

The below are proved in Sec. C,

$$\left[\hat{a}, (\hat{a}^\dagger)^n\right] = n\,(\hat{a}^\dagger)^{n-1}\,, \tag{3}$$

$$\left[\hat{a}^\dagger, \hat{a}^n\right] = -\,n\,\hat{a}^{n-1}\,, \tag{4}$$

$$\left[\hat{a}, \mathrm{e}^{\alpha\,\hat{a}^\dagger}\right] = \alpha\,\mathrm{e}^{\alpha\,\hat{a}^\dagger}\,, \tag{5}$$

$$\left[\hat{a}^\dagger, \mathrm{e}^{-\alpha^*\,\hat{a}}\right] = \alpha^*\,\mathrm{e}^{-\alpha^*\,\hat{a}}\,. \tag{6}$$

3 Vacuum state

The vacuum state is defined by

$$\hat{a}\;|0\rangle = 0\,.$$

3.1 Expectations

In position-momentum phase-space,

$$\langle\hat{x}\rangle = \langle 0\,|\,\hat{x}\,|\,0\rangle = 0\,, \qquad \Delta\hat{x} = \sqrt{\langle 0\,|\,\hat{x}^2\,|\,0\rangle - \langle 0\,|\,\hat{x}\,|\,0\rangle^2} = \sqrt{\frac{\hbar}{2\,m\,\Omega}}\,,$$

$$\langle\hat{p}\rangle = \langle 0\,|\,\hat{p}\,|\,0\rangle = 0\,, \qquad \Delta\hat{p} = \sqrt{\langle 0\,|\,\hat{p}^2\,|\,0\rangle - \langle 0\,|\,\hat{p}\,|\,0\rangle^2} = \sqrt{\frac{\hbar\,m\,\Omega}{2}}\,.$$

The vacuum state therefore minimizes the uncertainty $\Delta\hat{x}\,\Delta\hat{p} = \hbar/2$.

The vacuum standard deviations are known as the zero point fluctuations,

$$x_{\mathrm{ZPF}} = \sqrt{\frac{\hbar}{2\,m\,\Omega}}\,, \qquad p_{\mathrm{ZPF}} = \sqrt{\frac{\hbar\,m\,\Omega}{2}}\,.$$

4 Fock states

Fock states are eigenstates of the number operator, and energy eigenstates,

$$\underset{\text{number}}{\hat{n}} = \hat{a}^\dagger \, \hat{a} \ ,$$

$$\hat{n} \ |n\rangle = n \ |n\rangle \ .$$

They can be constructed with

$$\begin{aligned} |n\rangle &= \frac{1}{\sqrt{n!}} \left(\hat{a}^\dagger\right)^n |0\rangle \ , \\ \left(\hat{a}^\dagger\right)^m |n\rangle &= \sqrt{\frac{(n+m)!}{n!}} \ |n+m\rangle \ , \\ (\hat{a})^m |n\rangle &= \sqrt{\frac{n!}{(n-m)!}} \ |n-m\rangle \quad : \quad m \leq n \ , \\ \hat{a} \ |0\rangle &= 0 \ . \end{aligned}$$

4.1 Expectations

In position-momentum phase-space,

$$\langle \hat{x} \rangle = \langle n \,|\, \hat{x} \,|\, n \rangle = 0 \ , \qquad \Delta\hat{x} = \sqrt{\langle n \,|\, \hat{x}^2 \,|\, n \rangle - \langle n \,|\, \hat{x} \,|\, n \rangle^2} = \sqrt{\frac{\hbar}{m\,\Omega}\left(n + \frac{1}{2}\right)} \ ,$$

$$\langle \hat{p} \rangle = \langle n \,|\, \hat{p} \,|\, n \rangle = 0 \ , \qquad \Delta\hat{p} = \sqrt{\langle n \,|\, \hat{p}^2 \,|\, n \rangle - \langle n \,|\, \hat{p} \,|\, n \rangle^2} = \sqrt{\hbar\, m\, \Omega\left(n + \frac{1}{2}\right)} \ .$$

4.2 Wavefunctions

The wavefunction for Fock states in the position basis, $\psi_n(x) = \langle x \mid n \rangle$, can be found by solving the time-independent Schrödinger equation,

$$\left\langle x \,\middle|\, \hat{H} \,\middle|\, n \right\rangle = \left\langle x \,\middle|\, -\frac{\hbar^2}{2\,m} \frac{\partial^2}{\partial x^2} + \frac{m\,\Omega^2}{2}\, x^2 \,\middle|\, n \right\rangle = \left\langle x \,\middle|\, \hbar\,\Omega\left(n + \frac{1}{2}\right) \middle|\, n \right\rangle$$

$$\Rightarrow \quad -\frac{\hbar^2}{2\,m} \frac{\partial^2}{\partial x^2} \psi_n(t) + \frac{m\,\Omega^2}{2}\, x^2\, \psi_n(x) - \hbar\,\Omega\left(n + \frac{1}{2}\right) \psi_n(x) = 0 \ .$$

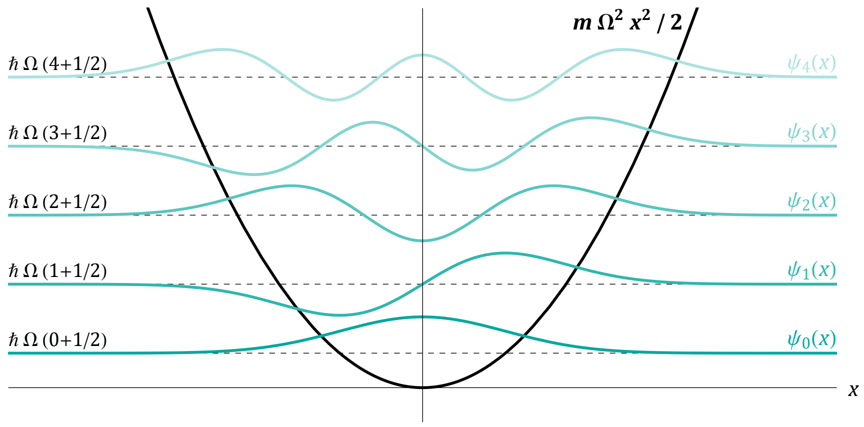

Figure 1: Quantum simple harmonic oscillator Fock state wavefunction plots.

Solutions to this differential equation are parabolic cylinder functions, of which only one with non-negative integer n is integrable, and can be related to Hermite polynomials [1],

$$\begin{aligned}\langle x \,|\, n\rangle = \psi_n(x) &= \frac{1}{\sqrt{n!}}\left(\frac{m\,\Omega}{\pi\,\hbar}\right)^{\frac{1}{4}} D_n\!\left[\sqrt{\frac{2\,m\,\Omega}{\hbar}}\,x\right] \qquad : \quad n \in \mathbb{N}\,, \\ &= \frac{1}{\sqrt{2^n\,n!}}\left(\frac{m\,\Omega}{\pi\,\hbar}\right)^{\frac{1}{4}} \mathrm{e}^{-\frac{m\,\Omega}{2\,\hbar}\,x^2}\, H_n\!\left[\sqrt{\frac{m\,\Omega}{\hbar}}\,x\right]\end{aligned}$$

with $\int_{-\infty}^{\infty} |\psi_n(x)|^2 \,\mathrm{d}x = 1$, where

$$\underset{\text{(physicist's) Hermite polynomial}}{H_n(x)} = (-1)^n\,\mathrm{e}^{x^2}\,\frac{\mathrm{d}^n}{\mathrm{d}x^n}\,\mathrm{e}^{-x^2}\,,$$

$$\begin{aligned}\underset{\text{parabolic cylinder}}{D_n(x)} = \frac{2^{\frac{n}{2}}}{\sqrt{\pi}}\,\mathrm{e}^{-\frac{x^2}{4}}\Bigg(&\cos\!\left[\frac{\pi\,n}{2}\right]\Gamma\!\left[\frac{n+1}{2}\right]{}_1F_1\!\left[-\frac{n}{2};\frac{1}{2};\frac{x^2}{2}\right] \\ &+\sqrt{2}\,x\,\sin\!\left[\frac{\pi\,n}{2}\right]\Gamma\!\left[\frac{n}{2}+1\right]{}_1F_1\!\left[\frac{1-n}{2};\frac{3}{2};\frac{x^2}{2}\right]\Bigg)\,.\end{aligned}$$

Plots of the first five wavefunctions are shown in Fig. 1.

4.3 Unitary transforms

For $k \in \mathbb{R}$,

$$\mathrm{e}^{\mathrm{i}\,k\,\hat{a}^\dagger\,\hat{a}}\,\hat{a}\,\mathrm{e}^{-\mathrm{i}\,k\,\hat{a}^\dagger\,\hat{a}} = \hat{a}\,\mathrm{e}^{-\mathrm{i}\,k} \qquad \mathrm{e}^{\mathrm{i}\,k\,\hat{a}^\dagger\,\hat{a}}\,\hat{a}^\dagger\,\mathrm{e}^{-\mathrm{i}\,k\,\hat{a}^\dagger\,\hat{a}} = \hat{a}^\dagger\,\mathrm{e}^{\mathrm{i}\,k}\,. \tag{7}$$

5 Coherent states

Coherent states are eigenstates of the annihilation operator,

$$\hat{a}\ |\alpha\rangle = \alpha\ |\alpha\rangle\ ,$$

where α is related to the expected number

$$\left\langle \alpha \,\middle|\, \hat{a}^\dagger\, \hat{a} \,\middle|\, \alpha \right\rangle = \ |\alpha|^2\ .$$

They can be constructed with the unitary displacement operator,

$$\underset{\text{displacement}}{\hat{D}[\alpha]} \quad = \quad \mathrm{e}^{\alpha\,\hat{a}^\dagger - \alpha^*\,\hat{a}}\ ,$$

$$|\alpha\rangle = \mathrm{e}^{\alpha\,\hat{a}^\dagger - \alpha^*\,\hat{a}}\ |0\rangle\ ,$$

which has the following properties

$$\hat{D}[-\alpha] = \hat{D}[\alpha]^\dagger\ ,$$
$$\hat{D}[\alpha]\,\hat{D}[\beta] = \mathrm{e}^{\mathrm{i}\,\Im[\alpha\,\beta^*]}\,\hat{D}[\alpha+\beta]\ .$$

The Kermack-McCrae identity is

$$\hat{D}[\alpha] = \mathrm{e}^{-|\alpha|^2/2}\,\mathrm{e}^{\alpha\,\hat{a}^\dagger}\,\mathrm{e}^{-\alpha^*\,\hat{a}} = \mathrm{e}^{|\alpha|^2/2}\,\mathrm{e}^{-\alpha^*\,\hat{a}}\,\mathrm{e}^{\alpha\,\hat{a}^\dagger}\ .$$

In the Fock basis,

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\ |n\rangle\ , \qquad (8)$$

which gives a Poisson probability distribution $|\langle n\,|\,\alpha\rangle|^2 = f_{\text{Poisson}}(n\,|\ |\alpha|^2)$.

5.1 Expectations

In position-momentum phase-space,

$$\langle\hat{x}\rangle = \langle\alpha\,|\,\hat{x}\,|\,\alpha\rangle = 2\sqrt{\frac{\hbar}{2\,m\,\Omega}}\,\Re[\alpha]\ , \qquad \Delta\hat{x} = \sqrt{\langle\alpha\,|\,\hat{x}^2\,|\,\alpha\rangle - \langle\alpha\,|\,\hat{x}\,|\,\alpha\rangle^2} = \sqrt{\frac{\hbar}{2\,m\,\Omega}}\ ,$$

$$\langle\hat{p}\rangle = \langle\alpha\,|\,\hat{p}\,|\,\alpha\rangle = 2\sqrt{\frac{\hbar\,m\,\Omega}{2}}\,\Im[\alpha]\ , \qquad \Delta\hat{p} = \sqrt{\langle\alpha\,|\,\hat{p}^2\,|\,\alpha\rangle - \langle\alpha\,|\,\hat{p}\,|\,\alpha\rangle^2} = \sqrt{\frac{\hbar\,m\,\Omega}{2}}\ .$$

Coherent states therefore minimize the uncertainty $\Delta\hat{x}\,\Delta\hat{p} = \hbar/2$.

For number,

$$\langle\hat{n}\rangle = \langle\alpha\,|\,\hat{n}\,|\,\alpha\rangle = \ |\alpha|^2\ , \qquad \Delta\hat{n} = \sqrt{\langle\alpha\,|\,\hat{n}^2\,|\,\alpha\rangle - \langle\alpha\,|\,\hat{n}\,|\,\alpha\rangle^2} = \ |\alpha|^2\ .$$

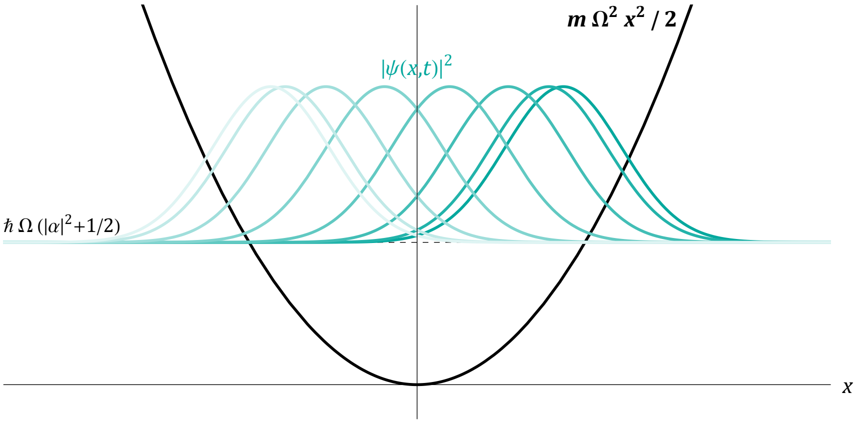

Figure 2: Quantum simple harmonic oscillator coherent state probability time evolution.

5.2 Wavefunctions

The wavefunction for a coherent state is

$$\langle x\,|\,\alpha\rangle = \left(\frac{m\,\Omega}{\pi\,\hbar}\right)^{\frac{1}{4}} \mathrm{e}^{\mathrm{i}\,\Im[\alpha]\left(\sqrt{\frac{2\,m\,\Omega}{\hbar}}\,x-\Re[\alpha]\right)-\left(\sqrt{\frac{m\,\Omega}{2\,\hbar}}\,x-\Re[\alpha]\right)^2} \tag{9}$$

$$= \left(\frac{m\,\Omega}{\pi\,\hbar}\right)^{\frac{1}{4}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\langle\hat{p}\rangle(x-\langle\hat{x}\rangle/2)-\frac{m\,\Omega}{2\,\hbar}(x-\langle\hat{x}\rangle)^2}\ .$$

Coherent states are not stationary (they are not eigenvectors of the Hamiltonian) and evolve according to

$$|\psi(t_0)\rangle = |\alpha_0\rangle \quad \Rightarrow \quad |\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}\,\Omega\,(t-t_0)/2}\,\left|\alpha_0\,\mathrm{e}^{-\mathrm{i}\,\Omega\,(t-t_0)}\right\rangle\ , \tag{10}$$

but remaining a coherent state.

Plots of a coherent state probability distribution evolving in time are shown in Fig. 2.

5.3 Unitary transforms

Using (5) and (6),

$$\hat{D}[\alpha]^\dagger\,\hat{a}\,\hat{D}[\alpha] = \hat{a} + \alpha\ ,$$

$$\hat{D}[\alpha]^\dagger\,\hat{a}^\dagger\,\hat{D}[\alpha] = \hat{a}^\dagger + \alpha^*\ ,$$

$$\hat{D}[\alpha]^\dagger\,\hat{a}^\dagger\,\hat{a}\,\hat{D}[\alpha] = \left(\hat{a}^\dagger + \alpha^*\right)\left(\hat{a} + \alpha\right)\ ,$$

$$\hat{D}\left[\alpha_0\,\mathrm{e}^{-\mathrm{i}\,\omega_\alpha\,t}\right]^\dagger \left(-\mathrm{i}\,\hbar\,\frac{\mathrm{d}}{\mathrm{d}t}\right) \hat{D}\left[\alpha_0\,\mathrm{e}^{-\mathrm{i}\,\omega_\alpha\,t}\right]$$
$$= -\mathrm{i}\,\hbar\,\frac{\mathrm{d}}{\mathrm{d}t} - \hbar\,\omega_\alpha\left(\alpha_0\,\mathrm{e}^{-\mathrm{i}\,\omega_\alpha\,t}\,\hat{a}^\dagger + \alpha_0^*\,\mathrm{e}^{\mathrm{i}\,\omega_\alpha\,t}\,\hat{a} + |\alpha_0|^2\right)\ . \tag{11}$$

6 Squeezed states

Squeezed vacuum states can be constructed with the unitary squeezing operator,

$$\underset{\text{squeezing}}{\hat{S}[\zeta]} = \mathrm{e}^{(\zeta^* \hat{a}\,\hat{a} - \zeta\,\hat{a}^\dagger \hat{a}^\dagger)/2} ,$$

$$|\zeta\rangle = \mathrm{e}^{(\zeta^* \hat{a}\,\hat{a} - \zeta\,\hat{a}^\dagger \hat{a}^\dagger)/2} \, |0\rangle ,$$

which has the following properties

$$\hat{S}[-\zeta] = \hat{S}[\zeta]^\dagger .$$

6.1 Expectations

In position-momentum phase-space,

$$\langle \hat{x} \rangle = \langle \zeta \,|\, \hat{x} \,|\, \zeta \rangle = 0 ,$$

$$\Delta\hat{x} = \sqrt{\langle \zeta \,|\, \hat{x}^2 \,|\, \zeta \rangle - \langle \zeta \,|\, \hat{x} \,|\, \zeta \rangle^2} = \sqrt{\frac{\hbar}{2\,m\,\Omega}} \sqrt{\cosh[2\,|\zeta|] - \Re\left[\sqrt{\zeta/\zeta^*}\right] \sinh[2\,|\zeta|]} , \tag{12}$$

$$\langle \hat{p} \rangle = \langle \zeta \,|\, \hat{p} \,|\, \zeta \rangle = 0 ,$$

$$\Delta\hat{p} = \sqrt{\langle \zeta \,|\, \hat{p}^2 \,|\, \zeta \rangle - \langle \zeta \,|\, \hat{p} \,|\, \zeta \rangle^2} = \sqrt{\frac{\hbar\,m\,\Omega}{2}} \sqrt{\cosh[2\,|\zeta|] + \Re\left[\sqrt{\zeta/\zeta^*}\right] \sinh[2\,|\zeta|]} ,$$

where $\Re\left[\sqrt{\zeta/\zeta^*}\right] = \cos[\angle\zeta]$, the cosine of the complex argument. For purely real squeezing,

$$\Delta\hat{x} = \sqrt{\frac{\hbar}{2\,m\,\Omega}}\, \mathrm{e}^{-\zeta} , \qquad \Delta\hat{p} = \sqrt{\frac{\hbar\,m\,\Omega}{2}}\, \mathrm{e}^{\zeta} \qquad : \quad \zeta \in \mathbb{R} .$$

6.2 Unitary transforms

Proved in Sec. C,

$$\hat{S}[\zeta]^\dagger \,\hat{a}\, \hat{S}[\zeta] = \hat{a}\, \cosh[|\zeta|] - \hat{a}^\dagger \frac{|\zeta|}{\zeta^*} \sinh[|\zeta|] , \tag{13}$$

$$\hat{S}[\zeta]^\dagger \,\hat{a}^\dagger\, \hat{S}[\zeta] = \hat{a}^\dagger \cosh[|\zeta|] - \hat{a}\, \frac{|\zeta|}{\zeta} \sinh[|\zeta|] .$$

7 Classicalization

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}(t) = \left[\hat{x}(t), \frac{-\mathrm{i}}{\hbar}\,\hat{H}\right] = \frac{\hat{p}(t)}{m}\ ,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{p}(t) = \left[\hat{p}(t), \frac{-\mathrm{i}}{\hbar}\,\hat{H}\right] = -m\,\Omega^2\,\hat{x}(t)\ ,$$

which are same as the classical Hamilton's equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}H}{\mathrm{d}p}\ , \qquad \frac{\mathrm{d}p}{\mathrm{d}t} = -\,\frac{\mathrm{d}H}{\mathrm{d}x}\ .$$

8 Spectral densities

For the annihilation operator,

$$\hat{a}(t) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\hat{H}\,t}\,\hat{a}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}\,t} \overset{(7)}{=} \hat{a}\,\mathrm{e}^{-\mathrm{i}\,\Omega\,t}\ , \tag{14}$$

$$S_{\hat{a}\,\hat{a}}(\omega) = \int_{-\infty}^{\infty} \left\langle \hat{a}(t)^{\dagger}\,\hat{a}(t+\tau)\right\rangle_t\,\mathrm{e}^{\mathrm{i}\,\omega\,\tau}\mathrm{d}\tau = \left\langle \hat{a}^{\dagger}\,\hat{a}\right\rangle\,2\,\pi\,\delta(\omega-\Omega)\ .$$

For position,

$$\hat{x}(t) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\hat{H}\,t}\,\hat{x}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}\,t} = \hat{x}\,\cos[\Omega\,t] + \frac{\hat{p}}{m\,\Omega}\,\sin[\Omega\,t]\ , \tag{15}$$

$$\begin{aligned} S_{\hat{x}\,\hat{x}}(\omega) &= \int_{-\infty}^{\infty} \left\langle \hat{x}(t)^{\dagger}\,\hat{x}(t+\tau)\right\rangle_t\,\mathrm{e}^{\mathrm{i}\,\omega\,\tau}\mathrm{d}\tau \\ &= \frac{\pi\,\hbar}{m\,\Omega}\Big(\left\langle \hat{a}^{\dagger}\,\hat{a}\right\rangle\,\big(\delta(\omega-\Omega)+\delta(\omega+\Omega)\big)+\delta(\omega-\Omega)\Big)\ . \end{aligned} \tag{16}$$

The total 'power' is

$$\left\langle \hat{x}^{\dagger}\,\hat{x}\right\rangle = \frac{1}{2\,\pi}\int_{-\infty}^{\infty} S_{\hat{x}\,\hat{x}}(\omega)\,\mathrm{d}\omega = \frac{\hbar}{m\,\Omega}\left(\left\langle \hat{a}^{\dagger}\,\hat{a}\right\rangle + \frac{1}{2}\right)\ ,$$

which for $n = 0$ gives ${x_{\mathrm{ZPF}}}^2$.

9 Heat bath with RWA

Coupling a harmonic oscillator to heat bath and applying the rotating-wave approximation (RWA) produces a framework for quantum input-output in terms of the resonator's square-root number amplitude a.

We can model an idealized bath with a set of independent harmonic oscillators

$$\hat{H}_{\text{bath}} = \sum_q \hbar\,\omega_q\,\hat{b}_q^\dagger\,\hat{b}_q\,, \qquad\qquad \left[\hat{b}_q, \hat{b}_{q'}^\dagger\right] = \delta_{q\,q'}\,,$$

and assume linear coupling

$$\hat{H}_{\text{int}} = \sum_q \hbar\left(\kappa_q\,\hat{a}^\dagger + \kappa_q^*\,\hat{a}\right)\left(\hat{b}_q^\dagger + \hat{b}_q\right) \approx \sum_q \hbar\left(\kappa_q^*\,\hat{a}\,\hat{b}_q^\dagger + \kappa_q\,\hat{a}^\dagger\,\hat{b}_q\right)\,, \tag{17}$$

whereby under the rotating-wave approximation, only the swapping interaction terms are relevant. κ_q has units time^{-1}. The complete Hamiltonian is

$$\hat{H} = \hat{H}_{\text{sys}}(\hat{a},\hat{a}^\dagger) + \underbrace{\sum_q \hbar\,\omega_q\,\hat{b}_q^\dagger\,\hat{b}_q}_{\hat{H}_{\text{bath}}} + \underbrace{\sum_q \hbar\left(\kappa_q^*\,\hat{a}\,\hat{b}_q^\dagger + \kappa_q\,\hat{a}^\dagger\,\hat{b}_q\right)}_{\hat{H}_{\text{int}}}\,,$$

where $\hat{H}_{\text{sys}}$ is some general system Hamiltonian constructed from $\hat{a}$ and $\hat{a}^\dagger$. For a simple harmonic oscillator $\hat{H}_{\text{sys}} = \hbar\,\Omega\left(\hat{a}^\dagger\,\hat{a} + \frac{1}{2}\right)$.

9.1 Equation of motion

The time-derivative of $\hat{b}_q(t)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{b}_q(t) \overset{(37)}{=} -\mathrm{i}\,\omega_q\,\hat{b}_q(t) - \mathrm{i}\,\kappa_q^*\,\hat{a}(t)\,,$$

which is a first-order differential equation with solution

$$\begin{aligned}\hat{b}_q(t) &= -\,\mathrm{i}\,\kappa_q^*\,\mathrm{e}^{-\mathrm{i}\,\omega_q\,t}\int^t \mathrm{e}^{\mathrm{i}\,\omega_q\,t'}\,\hat{a}(t')\,\mathrm{d}t' \\ &= -\,\mathrm{i}\,\kappa_q^*\int_{t_0}^t \mathrm{e}^{-\mathrm{i}\,\omega_q\,(t-t')}\,\hat{a}(t')\,\mathrm{d}t' + \mathrm{e}^{-\mathrm{i}\,\omega_q\,(t-t_0)}\,\hat{b}_q(t_0)\,.\end{aligned} \tag{18}$$

The time-derivative of $\hat{a}(t)$ is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}(t) &\overset{(37)}{=} \mathcal{U}\left[\left[\hat{a}, -\frac{\mathrm{i}}{\hbar}\hat{H}_{\text{sys}}\right]\right] - \sum_q \mathrm{i}\,\kappa_q\,\hat{b}_q(t) \\ &\overset{(18)}{=} \mathcal{U}\left[\left[\hat{a}, -\frac{\mathrm{i}}{\hbar}\hat{H}_{\text{sys}}\right]\right] - \sum_q \left|\kappa_q\right|^2 \int_{t_0}^t \mathrm{e}^{-\mathrm{i}\,\omega_q\,(t-t')}\,\hat{a}(t')\,\mathrm{d}t' - \sum_q \mathrm{i}\,\kappa_q\,\mathrm{e}^{-\mathrm{i}\,\omega_q\,(t-t_0)}\,\hat{b}_q(t_0)\,.\end{aligned}$$

Making the continuum approximation in frequency, $\sum_q \longmapsto \int_0^\infty \mathrm{d}\omega$ with $\kappa(\omega_q) = \kappa_q\sqrt{\text{time}}$ and $\hat{b}(\omega_q, t) = \hat{b}_q(t)\sqrt{\text{time}}$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}(t) &= \mathcal{U}\left[\left[\hat{a}, -\frac{\mathrm{i}}{\hbar}\hat{H}_{\text{sys}}\right]\right] \\ &\quad - \int_0^\infty \left|\kappa(\omega)\right|^2 \int_{t_0}^t \mathrm{e}^{-\mathrm{i}\,\omega\,(t-t')}\,\hat{a}(t')\,\mathrm{d}t'\,\mathrm{d}\omega - \int_0^\infty \mathrm{i}\,\kappa(\omega)\,\mathrm{e}^{-\mathrm{i}\,\omega\,(t-t_0)}\,\hat{b}(\omega, t_0)\,\mathrm{d}\omega\,.\end{aligned}$$

At this point, we need to make some approximations to continue. We will assume that our heat bath is Markovian - that is the correlation time between the bath and system, τ_{int}, is much less than the system's relaxation time. For a resonator, this means

$$\tau_{\text{int}} \sim \frac{2\pi}{\Omega} \ll \frac{2Q}{\Omega} = \tau_{\text{decay}} ,$$

which will be true for a sufficiently high quality factor resonator. In this case, coupling to the bath is only important over a narrow bandwidth around Ω, over which κ is approximately constant, and we can use a Markovian effective spectrum,

$$\int_0^\infty \mathrm{d}\omega \longmapsto \int_{-\infty}^\infty \mathrm{d}\omega \qquad\qquad \kappa(\omega) \approx \kappa(\Omega) = \kappa .$$

for the term containing $\hat{a}(t)$. Thus,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}(t) &\approx \mathcal{U}\left[\left[\hat{a}, -\frac{\mathrm{i}}{\hbar}\hat{H}_{\text{sys}}\right]\right] - |\kappa|^2 \int_{t_0}^{t}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\,\omega\,(t-t')}\,\mathrm{d}\omega\,\hat{a}(t')\,\mathrm{d}t' - \mathrm{i}\,\kappa \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\,\omega\,(t-t_0)}\,\hat{b}(\omega, t_0)\,\mathrm{d}\omega \\
&\overset{(40)}{=} \mathcal{U}\left[\left[\hat{a}, -\frac{\mathrm{i}}{\hbar}\hat{H}_{\text{sys}}\right]\right] - 2\pi\,|\kappa|^2 \int_{t_0}^{t} \delta(t-t')\,\hat{a}(t')\,\mathrm{d}t' - \mathrm{i}\,\kappa \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\,\omega\,(t-t_0)}\,\hat{b}(\omega, t_0)\,\mathrm{d}\omega \\
&= \mathcal{U}\left[\left[\hat{a}, -\frac{\mathrm{i}}{\hbar}\hat{H}_{\text{sys}}\right]\right] - \pi\,|\kappa|^2\,\hat{a}(t) - \sqrt{2\pi}\,\mathrm{i}\,\kappa\,\hat{b}_{\text{in}}(t) ,
\end{aligned}$$

where we have defined

$$\hat{b}_{\text{in}}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\,\omega\,(t-t_0)}\,\hat{b}(\omega, t_0)\,\mathrm{d}\omega , \qquad \left[\hat{b}_{\text{in}}(t), \hat{b}_{\text{in}}^\dagger(t')\right] = \delta(t-t') , \tag{19}$$

noting that $\hat{b}_{\text{in}}(t)$ has units of $\sqrt{\text{number/time}}$. The Markovian approximation introduced $\delta(t-t')$, removing the dependence of $\hat{a}(t)$ on earlier times from the bath interaction.

Defining $\gamma = -2\pi\,\kappa^2$, such that γ has units of time^{-1}, we get the quantum Langevin equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}(t) = \mathcal{U}\left[\left[\hat{a}, -\frac{\mathrm{i}}{\hbar}\hat{H}_{\text{sys}}\right]\right] - \frac{|\gamma|}{2}\hat{a}(t) - \sqrt{\gamma}\,\hat{b}_{\text{in}}(t) . \tag{20}$$

If we consider some $t_1 > t$, then (18) can be reformulated as

$$\hat{b}_q(t) = \mathrm{i}\,\kappa_q^* \int_t^{t_1} \mathrm{e}^{-\mathrm{i}\,\omega_q\,(t-t')}\,\hat{a}(t')\,\mathrm{d}t' + \mathrm{e}^{\mathrm{i}\,\omega_q\,(t_1-t)}\,\hat{b}_q(t_1) ,$$

and the time reversed quantum Langevin equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}(t) = \mathcal{U}\left[\left[\hat{a}, -\frac{\mathrm{i}}{\hbar}\hat{H}_{\text{sys}}\right]\right] + \frac{|\gamma|}{2}\hat{a}(t) - \sqrt{\gamma}\,\hat{b}_{\text{out}}(t) , \tag{21}$$

with

$$\hat{b}_{\text{out}}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\,\omega\,(t_1-t)}\,\hat{b}(\omega, t_1)\,\mathrm{d}\omega , \qquad \left[\hat{b}_{\text{out}}(t), \hat{b}_{\text{out}}^\dagger(t')\right] = \delta(t-t') . \tag{22}$$

(23)

Comparing (20) with (21) we obtain

$$\hat{b}_{\text{out}}(t) - \hat{b}_{\text{in}}(t) = \sqrt{\gamma^*}\,\hat{a}(t)\,.$$

For $\hat{H}_{\text{sys}} = \hbar\,\Omega\left(\hat{a}^\dagger\,\hat{a} + \frac{1}{2}\right)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}(t) = -\mathrm{i}\,\Omega\,\hat{a}(t) - \frac{|\gamma|}{2}\hat{a}(t) - \sqrt{\gamma}\,\hat{b}_{\text{in}}(t)\,, \tag{24}$$

$$\hat{b}_{\text{out}}(t) = \hat{b}_{\text{in}}(t) + \sqrt{\gamma^*}\,\hat{a}(t)\,.$$

Deriving equations for other operators,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}^\dagger(t) = \mathrm{i}\,\Omega\,\hat{a}^\dagger(t) - \frac{|\gamma|}{2}\hat{a}^\dagger(t) - \sqrt{\gamma^*}\,\hat{b}^\dagger_{\text{in}}(t)\,,$$

$$\hat{b}^\dagger_{\text{out}}(t) = \hat{b}^\dagger_{\text{in}}(t) + \sqrt{\gamma}\,\hat{a}^\dagger(t)\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}(t) = \frac{\hat{p}(t)}{m} - \frac{|\gamma|}{2}\hat{x}(t) - \sqrt{\frac{\hbar}{2\,m\,\Omega}}\left(\sqrt{\gamma^*}\,\hat{b}^\dagger_{\text{in}}(t) + \sqrt{\gamma}\,\hat{b}_{\text{in}}(t)\right)\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{p}(t) = -m\,\Omega^2\,\hat{x}(t) - \frac{|\gamma|}{2}\hat{p}(t) - \mathrm{i}\sqrt{\frac{\hbar\,m\,\Omega}{2}}\left(\sqrt{\gamma^*}\,\hat{b}^\dagger_{\text{in}}(t) - \sqrt{\gamma}\,\hat{b}_{\text{in}}(t)\right)\,.$$

9.2 Ports

Multiple heat baths can be used to model different independent ports to an interposed system. For a set of ports, P, with couplings γ_p,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}(t) = \mathcal{U}\left[\!\left[\hat{a}, -\frac{\mathrm{i}}{\hbar}\hat{H}_{\text{sys}}\right]\!\right] - \frac{|\gamma|}{2}\hat{a}(t) - \sum_{p\in P}\sqrt{\gamma_p}\,\hat{b}^{\text{in}}_p(t)\,, \qquad |\gamma| = \sum_{p\in P}\left|\gamma_p\right|\,,$$

$$\hat{b}^{\text{out}}_p(t) = \hat{b}^{\text{in}}_p(t) + \sqrt{\gamma^*_p}\,\hat{a}(t) \quad : \; p \in P\,.$$

9.3 Fourier transform

The Fourier transforms (see Sec. B) of a simple harmonic oscillator, (24), are

$$\hat{a}(\omega) = \frac{\sqrt{\gamma}\,\hat{b}_{\text{in}}(\omega)}{\mathrm{i}\,(\omega-\Omega) - \frac{|\gamma|}{2}} = \frac{\sqrt{\gamma}\,\hat{b}_{\text{out}}(\omega)}{\mathrm{i}\,(\omega-\Omega) + \frac{|\gamma|}{2}}$$

$$\hat{a}^\dagger(\omega) = \frac{\sqrt{\gamma^*}\,\hat{b}^\dagger_{\text{in}}(\omega)}{\mathrm{i}\,(\omega+\Omega) - \frac{|\gamma|}{2}} = \frac{\sqrt{\gamma^*}\,\hat{b}^\dagger_{\text{out}}(\omega)}{\mathrm{i}\,(\omega+\Omega) + \frac{|\gamma|}{2}}$$

Or, for multiple ports is

$$\hat{a}(\omega) = \frac{\sum\limits_{p\in P} \sqrt{\gamma_p}\, \hat{b}_p^{\text{in}}(\omega)}{\mathrm{i}\,(\omega-\Omega) - \frac{|\gamma|}{2}} = \frac{\sum\limits_{p\in P} \sqrt{\gamma_p}\, \hat{b}_p^{\text{out}}(\omega)}{\mathrm{i}\,(\omega-\Omega) + \frac{|\gamma|}{2}} \,, \qquad \begin{aligned} \hat{b}_p^{\text{in}}(\omega) &= \mathrm{e}^{\mathrm{i}\,\omega\, t_0}\, \hat{b}_p(\omega, t_0) \,, \\ \hat{b}_p^{\text{out}}(\omega) &= \mathrm{e}^{\mathrm{i}\,\omega\, t_1}\, \hat{b}_p(\omega, t_1) \,. \end{aligned} \tag{25}$$

Note that $\hat{a}(t)$ has units $\sqrt{\text{number}}$, and $\hat{a}(\omega)$ has units $\sqrt{\text{number}} \times$ time.

9.4 Fock-state bath

From the definition (19) of $\hat{b}_{\text{in}}(t)$ and $\hat{b}_{\text{out}}(t)$, if the baths are in Fock states (e.g. thermal or vacuum), the input and output spectral densities are

$$\begin{aligned} \underset{\text{Fock}}{S_{\hat{b}_{\text{in}}\hat{b}_{\text{in}}}}(\omega) &= n(\omega, t_0) \,, & \underset{\text{Fock}}{S_{\hat{b}_{\text{in}}^\dagger \hat{b}_{\text{in}}^\dagger}}(\omega) &= n(-\omega, t_0) + 1 \,, \\ \underset{\text{Fock}}{S_{\hat{b}_{\text{out}}\hat{b}_{\text{out}}}}(\omega) &= n(\omega, t_1) \,, & \underset{\text{Fock}}{S_{\hat{b}_{\text{out}}^\dagger \hat{b}_{\text{out}}^\dagger}}(\omega) &= n(-\omega, t_1) + 1 \,. \end{aligned} \tag{26}$$

If we split a coherent signal into classical and quantum-fluctuation parts, $\hat{b}_{\text{io}}(t) = \bar{b}\, \mathrm{e}^{-\mathrm{i}\,\omega_L\, t} + \delta\hat{b}_{\text{io}}(t)$, the number operator is

$$\hat{b}_{\text{io}}^\dagger\, \hat{b}_{\text{io}}(t) = \left|\bar{b}\right|^2 + \bar{b}\, \mathrm{e}^{-\mathrm{i}\,\omega_L\, t}\, \delta\hat{b}_{\text{io}}(t) + \bar{b}^*\, \mathrm{e}^{\mathrm{i}\,\omega_L\, t}\, \delta\hat{b}_{\text{io}}^\dagger(t) + \mathcal{O}\!\left[\delta\hat{b}_{\text{io}}^\dagger\, \delta\hat{b}_{\text{io}}\right] .$$

Using the vacuum fluctuations ($n = 0$) spectral densities from (26), the number spectral density is

$$S_{(\hat{b}_{\text{io}}^\dagger\, \hat{b}_{\text{io}})(\hat{b}_{\text{io}}^\dagger\, \hat{b}_{\text{io}})}(\omega) = 2\,\pi\, \delta(\omega)\, \left|\bar{b}\right|^4 + \left|\bar{b}\right|^2 \,,$$

which is the spectral density for $\hat{b}_{\text{io}}^\dagger\, \hat{b}_{\text{io}} \sim \text{Poisson}\!\left[\left|\bar{b}\right|^2\right]$, i.e. shot noise.

9.5 Thermal bath

We can derive the thermal spectrum for a quantum simple harmonic oscillator by analyzing it in equilibrium with a thermal bath. Our Hamiltonian is

$$\hat{H} = \underbrace{\hbar\,\Omega \left(\hat{a}^\dagger\, \hat{a} + \frac{1}{2}\right) + \sum_q \hbar\, \omega_q\, \hat{b}_q^\dagger\, \hat{b}_q}_{\hat{H}_0} + \underbrace{\sum_q \hbar \left(\kappa_q^*\, \hat{a}\, \hat{b}_q^\dagger + \kappa_q\, \hat{a}^\dagger\, \hat{b}_q\right)}_{\hat{H}_{\text{int}}} ,$$

and we will denote the time-evolution operator for $\hat{H}_0$ as $\hat{U}_0(t)$, such that

$$\mathrm{i}\,\hbar\, \frac{\mathrm{d}}{\mathrm{d}t} \hat{U}_0(t) = \hat{H}_0\, \hat{U}_0(t) \,.$$

Moving to the interaction picture we get the unitary transform, new Hamiltonian, and time-evolution,

$$|\psi_{\text{I}}(t)\rangle = \hat{U}_0^\dagger(t)\, |\psi(t)\rangle \,, \qquad \hat{H}_{\text{I}} \overset{(38)}{=} \hat{U}_0^\dagger(t)\, \hat{H}_{\text{int}}\, \hat{U}_0(t) \,, \qquad |\psi_{\text{I}}(t)\rangle = \hat{U}_{\text{I}}(t - t_0)\, |\psi_{\text{I}}(t_0)\rangle \,.$$

Now,

$$\mathrm{i}\,\hbar\,\frac{\mathrm{d}}{\mathrm{d}t}\hat{U}_{\mathrm{I}}(t)=\hat{H}_{\mathrm{I}}(t)\,\hat{U}_{\mathrm{I}}(t)\quad\Rightarrow\quad \hat{U}_{\mathrm{I}}(t)=\hat{U}_{\mathrm{I}}(t_1)-\frac{\mathrm{i}}{\hbar}\int_{t_1}^{t}\hat{H}_{\mathrm{I}}(t')\,\hat{U}_{\mathrm{I}}(t')\,\mathrm{d}t'\ ,$$

$$\Rightarrow\quad \hat{U}_{\mathrm{I}}(t-t_0)=1-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}_{\mathrm{I}}(t'-t_0)\,\hat{U}_{\mathrm{I}}(t'-t_0)\,\mathrm{d}t'\ ,$$

by integrating, the fundamental theorem of calculus, and $\hat{U}_{\mathrm{I}}(0)=1$. Applying this to $|\psi_{\mathrm{I}}(t_0)\rangle$ we can create a Dyson expansion

$$\begin{aligned}|\psi_{\mathrm{I}}(t)\rangle &= |\psi_{\mathrm{I}}(t_0)\rangle-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}_{\mathrm{I}}(t'-t_0)\,\left|\psi_{\mathrm{I}}(t')\right\rangle\,\mathrm{d}t'\\
&= |\psi_{\mathrm{I}}(t_0)\rangle-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}_{\mathrm{I}}(t'-t_0)\,|\psi_{\mathrm{I}}(t_0)\rangle\,\mathrm{d}t'\\
&\qquad+\left(\frac{\mathrm{i}}{\hbar}\right)^2\int_{t_0}^{t}\int_{t_0}^{t_1}\hat{H}_{\mathrm{I}}(t'-t_0)\,\hat{H}_{\mathrm{I}}(t''-t_0)\,\left|\psi_{\mathrm{I}}(t'')\right\rangle\,\mathrm{d}t''\,\mathrm{d}t'\\
&= \cdots\ ,\end{aligned}$$

and for extremely short times get away with only using the first two terms in the second expansion.

Let us now split $|\psi\rangle=\left|\psi_{\mathrm{sys}},\,\psi_{\mathrm{bath}}\right\rangle$, and let $|\psi_{\mathrm{I}}(t_0)\rangle=|n,\psi_{\mathrm{bath}}\rangle$ where the oscillator has occupation n and the bath is in some state. The probability amplitude of the oscillator gaining, or losing, a quanta (and the bath being in an arbitrary state) after some short time t is

$$\begin{aligned}\left\langle n\pm1,\psi'_{\mathrm{bath}}\middle|\,\hat{U}_{\mathrm{I}}(t-t_0)\,\middle|n,\psi_{\mathrm{bath}}\right\rangle &= -\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\left\langle n\pm1,\psi'_{\mathrm{bath}}\middle|\,\hat{H}_{\mathrm{I}}(t'-t_0)\,\middle|n,\psi_{\mathrm{bath}}\right\rangle\,\mathrm{d}t'\\
= -\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}&\left\langle n\pm1,\psi'_{\mathrm{bath}}\middle|\,\hat{U}_0^\dagger(t'-t_0)\,\sum_q\hbar\left(\kappa_q^*\,\hat{a}\,\hat{b}_q^\dagger+\kappa_q\,\hat{a}^\dagger\,\hat{b}_q\right)\,\hat{U}_0(t'-t_0)\,\middle|n,\psi_{\mathrm{bath}}\right\rangle\,\mathrm{d}t'\ .\end{aligned}$$

Now, with

$$\hat{U}_0(t-t_0)=\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}_0\,(t-t_0)}=\mathrm{e}^{-\mathrm{i}\,\Omega\left(\hat{a}^\dagger\,\hat{a}+\frac{1}{2}\right)(t-t_0)}\prod_q\mathrm{e}^{-\mathrm{i}\,\omega_q\,\hat{b}_q^\dagger\,\hat{b}_q\,(t-t_0)}\ ,$$

and using (7), find

$$\begin{aligned}\hat{U}_0^\dagger(t-t_0)\,\hat{a}\,\hat{U}_0(t-t_0)&=\hat{a}\,\mathrm{e}^{-\mathrm{i}\,\Omega\,(t-t_0)}\ ,&\quad \hat{U}_0^\dagger(t-t_0)\,\hat{b}_q\,\hat{U}_0(t-t_0)&=\hat{b}_q\,\mathrm{e}^{-\mathrm{i}\,\omega_q\,(t-t_0)}\ ,\\
\hat{U}_0^\dagger(t-t_0)\,\hat{a}^\dagger\,\hat{U}_0(t-t_0)&=\hat{a}^\dagger\,\mathrm{e}^{\mathrm{i}\,\Omega\,(t-t_0)}\ ,&\quad \hat{U}_0^\dagger(t-t_0)\,\hat{b}_q^\dagger\,\hat{U}_0(t-t_0)&=\hat{b}_q^\dagger\,\mathrm{e}^{\mathrm{i}\,\omega_q\,(t-t_0)}\ .\end{aligned}$$

Substituting these in,

$$
\begin{aligned}
&\left\langle n \pm 1, \psi'_{\text{bath}} \middle| \hat{U}_{\text{I}}(t-t_0) \middle| n, \psi_{\text{bath}} \right\rangle \\
&\qquad = -\mathrm{i} \int_{t_0}^{t} \mathrm{e}^{-\mathrm{i}\,\Omega\,(t'-t_0)} \, \langle n \pm 1 \,|\, \hat{a} \,|\, n \rangle \left\langle \psi'_{\text{bath}} \middle| \sum_q \kappa_q^* \, \mathrm{e}^{\mathrm{i}\,\omega_q\,(t'-t_0)} \, \hat{b}_q^\dagger \middle| \psi_{\text{bath}} \right\rangle \mathrm{d}t' \\
&\qquad\quad -\mathrm{i} \int_{t_0}^{t} \mathrm{e}^{\mathrm{i}\,\Omega\,(t'-t_0)} \, \left\langle n \pm 1 \middle| \hat{a}^\dagger \middle| n \right\rangle \left\langle \psi'_{\text{bath}} \middle| \sum_q \kappa_q \, \mathrm{e}^{-\mathrm{i}\,\omega_q\,(t'-t_0)} \, \hat{b}_q \middle| \psi_{\text{bath}} \right\rangle \mathrm{d}t' \\
&\qquad = \langle n \pm 1 \,|\, \hat{a} \,|\, n \rangle \, \sqrt{\gamma^*} \int_{t_0}^{t} \mathrm{e}^{-\mathrm{i}\,\Omega\,(t'-t_0)} \left\langle \psi'_{\text{bath}} \middle| \hat{b}_{\text{in}}^\dagger(t') \middle| \psi_{\text{bath}} \right\rangle \mathrm{d}t' \\
&\qquad\quad + \left\langle n \pm 1 \middle| \hat{a}^\dagger \middle| n \right\rangle \sqrt{\gamma} \int_{t_0}^{t} \mathrm{e}^{\mathrm{i}\,\Omega\,(t'-t_0)} \left\langle \psi'_{\text{bath}} \middle| \hat{b}_{\text{in}}(t') \middle| \psi_{\text{bath}} \right\rangle \mathrm{d}t' \ ,
\end{aligned}
$$

where we have used the continuum approximation and (19).

The probability of the oscillator gaining a quanta (and the bath being in any state) is thus

$$
\begin{aligned}
\mathbb{P}_{n\to n+1} &= \left| \left\langle n+1 \middle| \hat{U}_{\text{I}}(t-t_0) \middle| n \right\rangle \right|^2 = \sum_{\psi'_{\text{bath}}} \left| \left\langle n+1, \psi'_{\text{bath}} \middle| \hat{U}_{\text{I}}(t-t_0) \middle| n, \psi_{\text{bath}} \right\rangle \right|^2 \\
&= (n+1)\ |\gamma| \int_{t_0}^{t} \int_{t_0}^{t} \mathrm{e}^{\mathrm{i}\,\Omega\,(t''-t')} \\
&\qquad \times \sum_{\psi'_{\text{bath}}} \left\langle \psi_{\text{bath}} \middle| \hat{b}_{\text{in}}^\dagger(t') \middle| \psi'_{\text{bath}} \right\rangle \left\langle \psi'_{\text{bath}} \middle| \hat{b}_{\text{in}}(t'') \middle| \psi_{\text{bath}} \right\rangle \mathrm{d}t' \, \mathrm{d}t'' \\
&= (n+1)\ |\gamma| \int_{t_0}^{t} \int_{t_0}^{t} \mathrm{e}^{\mathrm{i}\,\Omega\,(t''-t')} \left\langle \hat{b}_{\text{in}}^\dagger(t') \, \hat{b}_{\text{in}}(t'') \right\rangle \mathrm{d}t' \, \mathrm{d}t'' \\
&= (n+1)\ |\gamma| \int_{t_0}^{t} \int_{t_0-t'}^{t-t'} \mathrm{e}^{\mathrm{i}\,\Omega\,\tau} \left\langle \hat{b}_{\text{in}}^\dagger(t') \, \hat{b}_{\text{in}}(t'+\tau) \right\rangle \mathrm{d}\tau \, \mathrm{d}t' \quad : \quad t'' = t' + \tau \ .
\end{aligned}
$$

Similarly, the probability of the oscillator losing a quanta is

$$
\begin{aligned}
\mathbb{P}_{n\to n-1} &= \left| \left\langle n-1 \middle| \hat{U}_{\text{I}}(t-t_0) \middle| n \right\rangle \right|^2 \\
&= n\ |\gamma| \int_{t_0}^{t} \int_{t_0}^{t} \mathrm{e}^{-\mathrm{i}\,\Omega\,(t''-t')} \left\langle \hat{b}_{\text{in}}(t') \, \hat{b}_{\text{in}}^\dagger(t'') \right\rangle \mathrm{d}t' \, \mathrm{d}t'' \\
&= n\ |\gamma| \int_{t_0}^{t} \int_{t_0-\prime}^{t-t'} \mathrm{e}^{-\mathrm{i}\,\Omega\,\tau} \left\langle \hat{b}_{\text{in}}(t') \, \hat{b}_{\text{in}}^\dagger(t'+\tau) \right\rangle \mathrm{d}\tau \, \mathrm{d}t' \quad : \quad t'' = t' + \tau \ .
\end{aligned}
$$

Using the Markovian approximation to assume our bath auto-correlations are extremely short, we can expand the τ integration to $(-\infty, \infty)$, obtaining

$$\mathbb{P}_{n\to n+1} \approx (n+1)\,|\gamma| \int_{t_0}^{t}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\,\Omega\,\tau} \left\langle \hat{b}_{\mathrm{in}}^{\dagger}(t')\,\hat{b}_{\mathrm{in}}(t'+\tau)\right\rangle \mathrm{d}\tau\,\mathrm{d}t'$$

$$= (n+1)\,|\gamma|\,(t-t_0)\,S_{\hat{b}_{\mathrm{in}}\hat{b}_{\mathrm{in}}}(\Omega)\,,$$

and

$$\mathbb{P}_{n\to n-1} \approx n\,|\gamma| \int_{t_0}^{t}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\,\Omega\,\tau} \left\langle \hat{b}_{\mathrm{in}}(t')\,\hat{b}_{\mathrm{in}}^{\dagger}(t'+\tau)\right\rangle \mathrm{d}\tau\,\mathrm{d}t'$$

$$= n\,|\gamma|\,(t-t_0)\left(S_{\hat{b}_{\mathrm{in}}\hat{b}_{\mathrm{in}}}(\Omega)+1\right) \tag{27}$$

Computing the rates of increase and decrease then,

$$\nu_{n\to n+1} = \tfrac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}_{n\to n+1} = (n+1)\,|\gamma|\,S_{\hat{b}_{\mathrm{in}}\hat{b}_{\mathrm{in}}}(\Omega) = (n+1)\,\Gamma_{\uparrow}\,, \qquad \Gamma_{\uparrow} = |\gamma|\,S_{\hat{b}_{\mathrm{in}}\hat{b}_{\mathrm{in}}}(\Omega)\,,$$
$$\nu_{n\to n-1} = \tfrac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}_{n\to n-1} = n\,|\gamma|\left(S_{\hat{b}_{\mathrm{in}}\hat{b}_{\mathrm{in}}}(\Omega)+1\right) = n\,\Gamma_{\downarrow}\,, \qquad \Gamma_{\downarrow} = |\gamma|\left(S_{\hat{b}_{\mathrm{in}}\hat{b}_{\mathrm{in}}}(\Omega)+1\right).$$

The probability flux for a certain occupation level is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}_n &= \nu_{n-1\to n}\,\mathbb{P}_{n-1} + \nu_{n+1\to n}\,\mathbb{P}_{n+1} - \nu_{n\to n-1}\,\mathbb{P}_n - \nu_{n\to n+1}\,\mathbb{P}_n\\ &= n\,\Gamma_{\uparrow}\,\mathbb{P}_{n-1} + (n+1)\,\Gamma_{\downarrow}\,\mathbb{P}_{n+1} - n\,\Gamma_{\downarrow}\,\mathbb{P}_n - (n+1)\,\Gamma_{\uparrow}\,\mathbb{P}_n\,.\end{aligned}$$

The average occupation level, and rate of change, is

$$\bar{n} = \sum_{n=0}^{\infty} n\,\mathbb{P}_n\,, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\bar{n} = \sum_{n=0}^{\infty} n\,\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}_n = \Gamma_{\uparrow} - (\Gamma_{\downarrow}-\Gamma_{\uparrow})\,\bar{n}\,.$$

Note that $\Gamma_{\downarrow} - \Gamma_{\uparrow} = |\gamma|$ which we might expect from

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\hat{a}^{\dagger}(t)\,\hat{a}(t)\right) = -\,|\gamma|\;\hat{a}^{\dagger}(t)\,\hat{a}(t) - \sqrt{\gamma}\,\hat{a}^{\dagger}(t)\,\hat{b}_{\mathrm{in}}(t) - \sqrt{\gamma^{*}}\,\hat{a}(t)\,\hat{b}_{\mathrm{in}}^{\dagger}(t)\,.$$

In thermal equilibrium we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{n} = 0 \quad\Rightarrow\quad \frac{\Gamma_{\downarrow}}{\Gamma_{\uparrow}} = \frac{\bar{n}+1}{\bar{n}} \quad\Rightarrow\quad S_{\hat{b}_{\mathrm{in}}\hat{b}_{\mathrm{in}}}(\Omega) = \bar{n}\,,$$

or equivalently, we have detailed balance,

$$\mathbb{P}_n\,\nu_{n\to n+1} = \mathbb{P}_{n+1}\,\nu_{n+1\to n} \quad\Rightarrow\quad \frac{\Gamma_{\downarrow}}{\Gamma_{\uparrow}} = \frac{\mathbb{P}_n}{\mathbb{P}_{n+1}}\,.$$

This also tells us

$$\mathbb{P}_n = \mathbb{P}_0\left(\frac{\Gamma_{\uparrow}}{\Gamma_{\downarrow}}\right)^{n}, \qquad \mathbb{P}_0 = 1 - \frac{\Gamma_{\uparrow}}{\Gamma_{\downarrow}}\,.$$

Our resonator obeys quantum Bose-Einstein statistics in thermal equilibrium,

$$Z = \sum_{n=0}^{\infty} \mathrm{e}^{-\frac{\hbar\,\Omega\left(n+\frac{1}{2}\right)}{k_\mathrm{B}\,\mathbb{T}}} = \frac{1}{2}\,\mathrm{csch}\left[\frac{\hbar\,\Omega}{2\,k_\mathrm{B}\,\mathbb{T}}\right] \quad \Rightarrow \quad \mathbb{P}_n = \frac{\mathrm{e}^{-\frac{\hbar\,\Omega\left(n+\frac{1}{2}\right)}{k_\mathrm{B}\,\mathbb{T}}}}{Z}\,, \quad \bar{n} = \frac{1}{\mathrm{e}^{\frac{\hbar\,\Omega}{k_\mathrm{B}\,\mathbb{T}}}-1}\,,$$

with average energy

$$-\frac{\mathrm{d}}{\mathrm{d}(k_\mathrm{B}\,\mathbb{T})^{-1}}\log[Z] = \hbar\,\Omega\left(\bar{n}+\frac{1}{2}\right) = \frac{\hbar\,\Omega}{2}\,\coth\left[\frac{\hbar\,\Omega}{2\,k_\mathrm{B}\,\mathbb{T}}\right].$$

In the limit of $\hbar \to 0$ or $k_\mathrm{B}\,\mathbb{T} \gg \hbar\,\omega$ this energy goes to $k_\mathrm{B}\,\mathbb{T}$. Thus,

$$S_{\hat{b}_\mathrm{in}\hat{b}_\mathrm{in}}(\Omega) = \bar{n} = \frac{1}{\mathrm{e}^{\frac{\hbar\,\Omega}{k_\mathrm{B}\,\mathbb{T}}}-1}\,.$$

Using the Fourier transform, (25),

$$S_{\hat{a}\hat{a}}(\omega) = \frac{|\gamma|\;S_{\hat{b}_\mathrm{in}\hat{b}_\mathrm{in}}(\omega)}{\frac{|\gamma|^2}{4}+(\omega-\Omega)^2} \approx \frac{|\gamma|\;S_{\hat{b}_\mathrm{in}\hat{b}_\mathrm{in}}(\Omega)}{\frac{|\gamma|^2}{4}+(\omega-\Omega)^2} = \frac{|\gamma|\;\bar{n}}{\frac{|\gamma|^2}{4}+(\omega-\Omega)^2}\,,$$

when γ is sufficiently narrow such that $S_{\hat{b}_\mathrm{in}\hat{b}_\mathrm{in}}(\omega)$ is approximately constant near $\omega = \Omega$. This is a two-sided spectrum but not symmetric. The total 'power' is

$$|\bar{a}|^2 = \frac{1}{2\,\pi}\int_{-\infty}^{\infty} S_{\hat{a}\hat{a}}(\omega)\,\mathrm{d}\omega = \bar{n}\,.$$

Performing the analysis of quanta jump probabilities and detailed balance with the alternative ordering $\hat{b}_\mathrm{in}\,\hat{b}^\dagger_\mathrm{in}$ yields

$$S_{\hat{b}^\dagger_\mathrm{in}\hat{b}^\dagger_\mathrm{in}}(-\Omega) = \bar{n}+1\,,$$

and

$$S_{\hat{a}^\dagger\hat{a}^\dagger}(-\omega) = \frac{|\gamma|\;S_{\hat{b}^\dagger_\mathrm{in}\hat{b}^\dagger_\mathrm{in}}(-\omega)}{\frac{|\gamma|^2}{4}+(\omega-\Omega)^2} \approx \frac{|\gamma|\;S_{\hat{b}^\dagger_\mathrm{in}\hat{b}^\dagger_\mathrm{in}}(-\Omega)}{\frac{|\gamma|^2}{4}+(\omega-\Omega)^2} = \frac{|\gamma|\,(\bar{n}+1)}{\frac{|\gamma|^2}{4}+(\omega-\Omega)^2}\,.$$

Furthermore, the bath spectral densities are related by

$$S_{\hat{b}^\dagger_\mathrm{in}\hat{b}^\dagger_\mathrm{in}}(-\omega) = S_{\hat{b}_\mathrm{in}\hat{b}_\mathrm{in}}(\omega)+1\,. \tag{28}$$

The Markovian approximation for our heat bath dictates that the auto-correlation function is essentially $R_{\hat{b}_\mathrm{in}\hat{b}_\mathrm{in}}(\tau) = \delta(\tau)\,\sigma^2$, for some constant σ^2, which in turn sets $R_{\hat{b}^\dagger_\mathrm{in}\hat{b}^\dagger_\mathrm{in}}(\tau) = \delta(\tau)\,\left(\sigma^2+1\right)$. This implies the spectral densities are constant (white-noise), $S_{\hat{b}_\mathrm{in}\hat{b}_\mathrm{in}} = \sigma^2 = \bar{n}$ and $S_{\hat{b}^\dagger_\mathrm{in}\hat{b}^\dagger_\mathrm{in}} = \sigma^2+1 = \bar{n}+1$, matching the form of (26).

This analysis uses random fluctuations in cavity amplitude (occupation) rather than random fluctuations in a driving force as is common by other authors [2–6], and furthermore uses the rotating-wave approximation for bath coupling, (17). It is only valid for sufficiently high quality resonators, $Q \gtrsim 10$. Note that at high temperatures,

$$k_\mathrm{B}\,\mathbb{T} \gg \hbar\,\Omega \quad \Rightarrow \quad \bar{n} = \frac{1}{\mathrm{e}^{\frac{\hbar\,\Omega}{k_\mathrm{B}\,\mathbb{T}}} - 1} \approx \frac{k_\mathrm{B}\,\mathbb{T}}{\hbar\,\Omega}\,.$$

Within this rotating wave approximation, we get

$$\bar{S}_{\hat{x}\hat{x}}(\omega \approx \Omega) = \frac{\hbar\,|\gamma|\left(\bar{n} + \frac{1}{2}\right)}{2\,m\,\Omega\left((\omega - \Omega)^2 + \frac{|\gamma|^2}{4}\right)}\,, \tag{29}$$

where $\bar{S}(\omega) = \frac{1}{2}\left(S(\omega) + S(-\omega)\right)$ is the symmetrized spectral density.

If there are multiple (thermalized) ports, each provides $S_{\hat{b}_i^{\mathrm{in}}\hat{b}_i^{\mathrm{in}}}(\Omega) = \bar{n}$, and couples with γ_i, such that the total energy remains the same.

9.6 Without rotating wave

Without the rotating wave approximation [6**?**], the mechanical susceptibility is

$$\chi(\omega) = \frac{\hat{x}(\omega)}{\hat{F}(\omega)} = \frac{1}{m\,\left(\Omega^2 - \omega^2 - \mathrm{i}\,\omega\,\gamma\right)}\,, \tag{30}$$

and $S_{\hat{x}\hat{x}}(\omega) = \left|\chi(\omega)\right|^2\,S_{\hat{F}\hat{F}}(\omega)$. In terms of input operators, $\hat{F} = \sqrt{2\,\gamma}\,\hat{p}_{\mathrm{in}}$.

9.6.1 Fluctuation-dissipation

For a thermal force, the fluctuation-dissipation theorem [2] gives

$$S_{\hat{x}\hat{x}}(\{\omega, -\omega\}) = 2\,\hbar\left(\bar{n} + \{1, 0\}\right)\mathfrak{I}\left[\chi(\omega)\right]\,,$$

$$\bar{S}_{\hat{x}\hat{x}}(\omega) = 2\,\hbar\left(\bar{n} + \frac{1}{2}\right)\mathfrak{I}\left[\chi(\omega)\right]\,.$$

With

$$\left|\chi(\omega)\right|^2 = \frac{S_{\hat{x}\hat{x}}(\omega)}{S_{\hat{F}\hat{F}}(\omega)}\,, \qquad \frac{\mathfrak{I}\left[\chi(\omega)\right]}{\left|\chi(\omega)\right|^2} = m\,\omega\,\gamma\,,$$

we can solve

$$S_{\hat{F}\hat{F}}(\{\omega, -\omega\}) = 2\,\hbar\,m\,\omega\,\gamma\left(\bar{n} + \{1, 0\}\right)\,,$$

$$\bar{S}_{\hat{F}\hat{F}}(\omega) = 2\,\hbar\,m\,\omega\,\gamma\left(\bar{n} + \frac{1}{2}\right)\,.$$

The dissipation is

$$\gamma = \frac{{x_{\mathrm{ZPF}}}^2}{\hbar^2}\left(S_{\hat{F}\hat{F}}(\Omega) - S_{\hat{F}\hat{F}}(-\Omega)\right)\,.$$

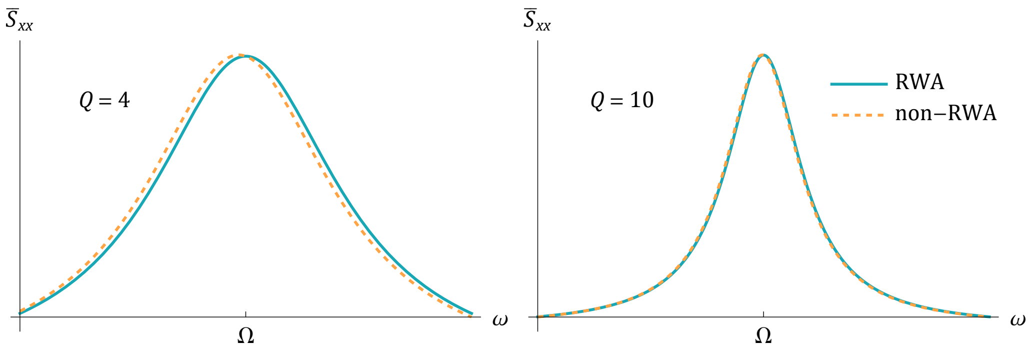

Figure 3: Comparison of $\bar{S}_{\hat{x}\hat{x}(\omega)}$ with and without the rotating wave approximation.

9.6.2 Comparison

Here, we have, using (30),

$$\bar{S}_{\hat{x}\hat{x}}(\omega \approx \Omega) = \frac{2\hbar\gamma\Omega\left(\bar{n}+\frac{1}{2}\right)}{m\left((\omega^2-\Omega^2)^2+\omega^2\gamma^2\right)}\,.$$

A qualitative comparison to the rotating wave approximation result, (29), is plotted in Fig. 3.

In both cases, the thermal input is

$$\bar{S}_{\hat{p}_{\rm in}\hat{p}_{\rm in}}(\Omega) = \hbar\, m\,\Omega\left(\bar{n}+\frac{1}{2}\right)\,.$$

10 Coherent drive

A forced harmonic oscillator can be realized be adding a term $F(t)\,\hat{x}$ to the Hamiltonian. In ladder operator formalism the most general expression is

$$\hat{H}_{\rm drive} = f(t)\,\hat{a}^\dagger + f(t)^*\,\hat{a}\,.$$

If our drive is coherent, then $f(t) \propto e^{-i\omega_L t}$ and our Hamiltonian is

$$\hat{H}(t) = \underbrace{\hbar\Omega\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)}_{\hat{H}_{\rm sys}} + \underbrace{\hbar L\,\hat{a}^\dagger\,e^{-i\omega_L t} + \hbar L^*\,\hat{a}\,e^{i\omega_L t}}_{\hat{H}_{\rm drive}}\,. \tag{31}$$

The Schödinger operator for (31) has coherent state eigenstates,

$$\left(\hat{H} - i\hbar\frac{d}{dt}\right)\left|\alpha_0\,e^{-i\omega_L t}\right\rangle = \hbar\left(\frac{\Omega}{2} - |\alpha_0|^2\,(\Omega-\omega_L)\right)\left|\alpha_0\,e^{-i\omega_L t}\right\rangle \quad : \quad \alpha_0 = \frac{-L}{\Omega-\omega_L}\,. \tag{32}$$

Driving on resonance without any loss will explode our resonator.

10.1 Equation of motion

The equation of motion for $\hat{a}$, including heat baths, is

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}(t) \overset{(37)}{=} -\mathrm{i}\,\Omega\,\hat{a}(t) - \frac{|\gamma|}{2}\,\hat{a}(t) - \mathrm{i}\,L\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t} - \sum_{p\in P}\sqrt{\gamma_p}\,\hat{b}_p^{\mathrm{in}}(t)\,. \tag{33}$$

If the drive contributes to loss, it can take the place of one of the ports,

$$\sqrt{\gamma_L}\,\hat{b}_\mathrm{L}^{\mathrm{in}}(t) = \mathrm{i}\,L\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t}\,.$$

Relating to input power, we deduce

$$P_\mathrm{L} = \hbar\,\omega_\mathrm{L}\,\left\langle \hat{b}_\mathrm{L}^{\mathrm{in}\,\dagger}(t)\,\hat{b}_\mathrm{L}^{\mathrm{in}}(t)\right\rangle \quad\Rightarrow\quad |L|^2 = |\gamma_\mathrm{L}|\,\frac{P_\mathrm{L}}{\hbar\,\omega_\mathrm{L}}\,.$$

10.2 Steady-state

With only a single coherent drive and $\hat{b}_{\mathrm{in}}(t) = 0$, a steady-population solution to (33) exists,

$$\hat{a}_{\mathrm{std}}(t) = \frac{-\mathrm{i}\,L\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t}}{\mathrm{i}\,(\Omega-\omega_\mathrm{L}) + \frac{|\gamma|}{2}} = \bar{a}_\mathrm{L}\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t}\,,$$

with $\hat{a}^\dagger_{\mathrm{std}}(t)\,\hat{a}_{\mathrm{std}}(t) = |\bar{a}_\mathrm{L}|^2$.

10.3 Rotating frame

The rotating frame can be used to take the coherent drive time dependence out of the Hamiltonian. Without heat baths, for unitary transform

$$\hat{U}_{\mathrm{rot}}(t) = \mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,(\hat{a}^\dagger\,\hat{a}+1/2)\,t}\,,\qquad |\psi_{\mathrm{rot}}(t)\rangle = \hat{U}^\dagger_{\mathrm{rot}}(t)\,|\psi(t)\rangle\,,$$

$$\hat{H}_{\mathrm{rot}} \overset{(38)}{=} \hbar\,(\Omega-\omega_\mathrm{L})\left(\hat{a}^\dagger\,\hat{a} + \frac{1}{2}\right) + \hbar\,L\,\hat{a}^\dagger + \hbar\,L^*\,\hat{a}\,.$$

If heat baths exist, they can be rotated together with minimal consequence,

$$\hat{U}_{\mathrm{rot}}(t) = \mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,(\hat{a}^\dagger\,\hat{a}+1/2)\,t}\prod_q \mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,\hat{b}_q^\dagger\,\hat{b}_q}\,,\qquad |\psi_{\mathrm{rot}}(t)\rangle = \hat{U}^\dagger_{\mathrm{rot}}(t)\,|\psi(t)\rangle\,,$$

$$\hat{H}_{\mathrm{rot}} \overset{(38)}{=} \hbar\,\Delta\Omega\left(\hat{a}^\dagger\,\hat{a} + \frac{1}{2}\right) + \hbar\,L\,\hat{a}^\dagger + \hbar\,L^*\,\hat{a} + \sum_q \hbar\,\Delta\omega_q\,\hat{b}_q^\dagger\,\hat{b}_q + \sum_q \hbar\left(\kappa_q^*\,\hat{a}\,\hat{b}_q^\dagger + \kappa_q\,\hat{a}^\dagger\,\hat{b}_q\right)\,,$$

with $\Delta\Omega = \Omega - \omega_\mathrm{L}$ and $\Delta\omega_q = \omega_q - \omega_\mathrm{L}$. The equation of motion is then

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}_{\mathrm{rot}}(t) \overset{(37)}{=} -\mathrm{i}\,\Delta\Omega\,\hat{a}_{\mathrm{rot}}(t) - \frac{|\gamma|}{2}\,\hat{a}_{\mathrm{rot}}(t) - \mathrm{i}\,L - \sum_{p\in P}\sqrt{\gamma_p}\,\hat{b}^{\mathrm{in}}_{\mathrm{rot},p}(t)\,.$$

10.4 Fluctuating frame

The fluctuating frame can be used to analyze fluctuations about a steady state. For a single drive,

$$\hat{U}_{\mathrm{fluc}}(t) = \hat{D}\big[\bar{a}_{\mathrm{L}}\,\mathrm{e}^{-\mathrm{i}\,\omega_{\mathrm{L}}\,t}\big]\,, \qquad \bar{a}_{\mathrm{L}} = \frac{-\mathrm{i}\,L}{\mathrm{i}\,(\Omega-\omega_{\mathrm{L}}) + \frac{|\gamma|}{2}}\,, \qquad |\psi_{\mathrm{fluc}}(t)\rangle = \hat{U}^{\dagger}_{\mathrm{fluc}}(t)\,|\psi(t)\rangle\,,$$

$$\begin{aligned}\hat{H}_{\mathrm{fluc}} \overset{(38)}{=} \hbar\,\Omega\left(\hat{a}^{\dagger}\,\hat{a} + \frac{1}{2}\right) + \hbar\,(\Omega-\omega_{\mathrm{L}})\,\left(\bar{a}_{\mathrm{L}}\,\mathrm{e}^{-\mathrm{i}\,\omega_{\mathrm{L}}\,t}\,\hat{a}^{\dagger} + \bar{a}^{*}_{\mathrm{L}}\,\mathrm{e}^{\mathrm{i}\,\omega_{\mathrm{L}}\,t}\,\hat{a} + |\bar{a}_{\mathrm{L}}|^{2}\right)\\ + \hbar\,\big(L\,\mathrm{e}^{-\mathrm{i}\,\omega_{\mathrm{L}}\,t}\,\hat{a}^{\dagger} + L\,\bar{a}^{*}_{\mathrm{L}} + L^{*}\,\mathrm{e}^{\mathrm{i}\,\omega_{\mathrm{L}}\,t}\,\hat{a} + L^{*}\,\bar{a}_{\mathrm{L}}\big)\\ + \sum_{q}\hbar\left(\omega_{q}\,\hat{b}^{\dagger}_{q}\,\hat{b}_{q} + \kappa^{*}_{q}\,\hat{b}^{\dagger}_{q}\,\hat{a} + \kappa^{*}_{q}\,\bar{a}_{\mathrm{L}}\,\mathrm{e}^{-\mathrm{i}\,\omega_{\mathrm{L}}\,t}\,\hat{b}^{\dagger}_{q} + \kappa_{q}\,\hat{b}_{q}\,\hat{a}^{\dagger} + \kappa_{q}\,\bar{a}^{*}_{\mathrm{L}}\,\mathrm{e}^{\mathrm{i}\,\omega_{\mathrm{L}}\,t}\,\hat{b}_{q}\right)\,.\end{aligned}$$

Following a similar procedure to Sec. 9.1, the equation of motion is then

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}_{\mathrm{fluc}}(t) = -\mathrm{i}\,\Omega\,\hat{a}_{\mathrm{fluc}}(t) - \frac{|\gamma|}{2}\,\hat{a}_{\mathrm{fluc}}(t) - \sum_{p\in P}\sqrt{\gamma_{p}}\,\hat{b}^{\mathrm{in}}_{p}(t)\,. \tag{34}$$

If there are multiple drives, it is possible for the simple harmonic oscillator to displace some or all of them out (See Ref. [7] Sec. I.6).

10.5 Noise-drive approximation

If white-noise is incident on our resonator, and γ is sufficiently narrow, we can make a crude approximation of the noise as a coherent drive at the resonator frequency.

For white-noise,

$$S_{\hat{b}_{\mathrm{in}}\hat{b}_{\mathrm{in}}}(\omega) = \sigma^{2}\,, \qquad \hat{b}_{\mathrm{in}}(\omega) = \sigma\,,$$

the filtered spectrum, and total 'power' is

$$S_{\hat{a}\hat{a}}(\omega) = \frac{|\gamma_{\mathrm{L}}|\;\sigma^{2}}{(\omega-\Omega)^{2} + \frac{|\gamma|^{2}}{4}}\,, \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{\hat{a}\hat{a}}(\omega)\,\mathrm{d}\omega = 2\pi\,\frac{|\gamma_{\mathrm{L}}|}{|\gamma|}\,\sigma^{2}\,.$$

For a coherent drive,

$$\sqrt{\gamma_{\mathrm{L}}}\,\hat{b}_{\mathrm{in}}(t) = \mathrm{i}\,L\,\mathrm{e}^{-\mathrm{i}\,\Omega\,t}\,, \qquad S_{\hat{b}_{\mathrm{in}}\hat{b}_{\mathrm{in}}}(\omega) = \frac{|L|^{2}}{|\gamma_{\mathrm{L}}|}\,2\pi\,\delta(\omega-\Omega)\,,$$

the filtered spectrum, and total 'power' is

$$S_{\hat{a}\hat{a}}(\omega) = \frac{2\pi\;|L|^{2}\;\delta(\omega-\Omega)}{(\omega-\Omega)^{2} + \frac{|\gamma|^{2}}{4}}\,, \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{\hat{a}\hat{a}}(\omega)\,\mathrm{d}\omega = 2\pi\,\frac{4\;|L|^{2}}{|\gamma|^{2}}\,.$$

Matching the filtered powers, we find

$$\mathrm{i}\,L = \frac{\sqrt{\gamma_{\mathrm{L}}}\,\sqrt{\gamma}\,\sigma}{2}\,.$$

A General quantum mechanics

A.1 Commutators

$$[\hat{A},\hat{B}] = \hat{A}\,\hat{B} - \hat{B}\,\hat{A}\ ,$$

$$[\hat{A},\hat{B}]^\dagger = [\hat{B}^\dagger,\hat{A}^\dagger] = -[\hat{A}^\dagger,\hat{B}^\dagger]\ ,$$

$$[\hat{A}\,\hat{B},\hat{C}] = \hat{A}\ [\hat{B},\hat{C}] + [\hat{A},\hat{C}]\ \hat{B}\ ,$$

$$[\hat{A}^m,\hat{B}^n] = \sum_{i=1}^{m}\sum_{j=1}^{n} \hat{A}^{m-i}\,\hat{B}^{n-j}\ [\hat{A},\hat{B}]\ \hat{B}^{j-1}\,\hat{A}^{i-1}\ ,$$

$$\left[\frac{\partial}{\partial t},\hat{A}(t)\right] = \left(\frac{\partial}{\partial t}\hat{A}(t)\right)\ . \tag{35}$$

A.2 Baker-Campbell-Hausdorff relations

For $\hat{U} = \mathrm{e}^{\hat{G}}$ and $\hat{U}^\dagger = \hat{U}^{-1} \Leftrightarrow \hat{G}^\dagger = -\hat{G}$,

$$\begin{aligned} \hat{U}^\dagger\,\hat{A}\,\hat{U} &= \mathrm{e}^{-\hat{G}}\,\hat{A}\,\mathrm{e}^{\hat{G}} \\ &= \hat{A} + [\hat{A},\hat{G}] + \frac{1}{2}\,[[\hat{A},\hat{G}]\,,\hat{G}] + \frac{1}{6}\,[[[\hat{A},\hat{G}]\,,\hat{G}]\,,\hat{G}] + \cdots \\ &= \sum_{i=0}^{\infty}\frac{1}{i!}\Big[\cdots\Big[\Big[\hat{A},\underbrace{\hat{G}\Big],\hat{G}\Big],\cdots,\hat{G}\Big]}_{i}\ . \end{aligned} \tag{36}$$

If $\hat{G}$ is infinitesimal, $\Delta\hat{A} = [\hat{A},\hat{G}] + \mathcal{O}[\hat{G}^2]$.

A.3 Schrödinger equation

The time-dependent Schrödinger equation is

$$\mathrm{i}\,\hbar\,\frac{\mathrm{d}}{\mathrm{d}t}\,|\psi(t)\rangle = \hat{H}(t)\,|\psi(t)\rangle\ .$$

A.4 Equations of motion

For a Hamiltonian, $\hat{H}(t)$, with unitary time-evolution operator

$$\hat{U}(t) = \hat{T}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int^t \hat{H}(t')\,\mathrm{d}t'}\ ,$$

let

$$\mathcal{U}\Big[\hat{A}\Big] = \hat{U}^\dagger(t)\,\hat{A}\,\hat{U}(t)\,,$$

such that the Heisenberg-picture operator is related to Shrödinger-picture operator as $\hat{A}_\mathrm{H} = \mathcal{U}[\hat{A}_\mathrm{S}]$. The Heisenberg equation of motion for an operator $\hat{A}$ is then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{U}\Big[\hat{A}\Big] = \Big[\mathcal{U}\Big[\hat{A}\Big], -\frac{\mathrm{i}}{\hbar}\mathcal{U}\Big[\hat{H}\Big]\Big] + \mathcal{U}\Big[\frac{\mathrm{d}}{\mathrm{d}t}\hat{A}\Big] = \mathcal{U}\Big[\Big[\hat{A}, -\frac{\mathrm{i}}{\hbar}\hat{H}\Big]\Big] + \mathcal{U}\Big[\frac{\mathrm{d}}{\mathrm{d}t}\hat{A}\Big]\,. \tag{37}$$

For a time-independent Hamiltonian,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{H} = 0 \quad \Rightarrow \quad \hat{U}(t) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}\,t}\,, \quad \mathcal{U}\Big[\hat{H}\Big] = \hat{H}\,.$$

A.5 Unitary transforms

Under a unitary transform

$$|\psi'(t)\rangle = \hat{U}^\dagger(t)\,|\psi(t)\rangle \quad \Rightarrow \quad \hat{H}'(t) = \hat{U}^\dagger(t)\Big(\hat{H}(t) - \mathrm{i}\,\hbar\,\frac{\mathrm{d}}{\mathrm{d}t}\Big)\,\hat{U}(t)\,, \tag{38}$$

such that

$$\mathrm{i}\,\hbar\,\frac{\mathrm{d}}{\mathrm{d}t}\,|\psi'(t)\rangle = \hat{H}'(t)\,|\psi'(t)\rangle\,.$$

This is equivalent to performing a change of basis.

A.6 Derivative operator

Within a quantum mechanics inner-product,

$$\hat{\partial}^\dagger = -\hat{\partial}\,. \tag{39}$$

B Fourier transforms

Define Fourier transforms on f as

$$\mathcal{F}[f][\omega] = f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\,\omega\,t} f(t)\,\mathrm{d}t \,,$$
$$\mathcal{F}^{-1}\big[\mathcal{F}[f]\big][t] = f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\,\omega\,t} f(\omega)\,\mathrm{d}\omega \,,$$

with

$$f(t)^* = f^*(t) \,, \qquad\qquad f(\omega)^* = f^*(-\omega) \,.$$

B.1 Dirac delta

The Dirac delta distribution obeys

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\,x\,y}\,\mathrm{d}x = 2\,\pi\,\delta[y] \,, \tag{40}$$

$$\int_{x_1<y}^{x_2>y} \delta[x-y]\,f[x]\,\mathrm{d}x = f[y] \,, \tag{41}$$

$$\int_{x_1<y}^{y} \delta[x-y]\,f[x]\,\mathrm{d}x = f[y]/2 \,, \tag{42}$$

for $y \in \mathbb{R}$. Dirac delta has units inverse to its argument.

B.2 Heaviside theta

The Heaviside step function is

$$\Theta[x] = \begin{cases} 0 & : \; x < 0 \\ \frac{1}{2} & : \; x = 0 \\ 1 & : \; x > 0 \end{cases} \,, \tag{43}$$

such that

$$\frac{\mathrm{d}}{\mathrm{d}x}\Theta[x] = \delta[x] \,, \qquad\qquad \Theta[x] = \int_{-\infty}^{x} \delta[y]\,\mathrm{d}y \,.$$

B.3 Other integrals

For $\{y, z\} \subset \mathbb{R}$,

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\,x\,y - z\,|x|}\,\mathrm{d}x = \frac{2\,z}{y^2 + z^2} \,.$$

C Proofs

Proof of (3).

$$\begin{aligned}\left[\hat{a},(\hat{a}^\dagger)^n\right] &= (\hat{a}^\dagger)^{n-1} + \hat{a}^\dagger\ \left[\hat{a},(\hat{a}^\dagger)^{n-1}\right]\\ &= \sum_{i=1}^{n}(\hat{a}^\dagger)^{n-1}\\ &= n\,(\hat{a}^\dagger)^{n-1}\ ,\end{aligned}$$

□

Proof of (4).

$$\begin{aligned}\left[\hat{a}^\dagger,\hat{a}^n\right] &= -\,\hat{a}^{n-1} + \hat{a}\ \left[\hat{a}^\dagger,\hat{a}^{n-1}\right]\\ &= -\sum_{i=1}^{n}\hat{a}^{n-1}\\ &= -\,n\,\hat{a}^{n-1}\ .\end{aligned}$$

□

Proof of (5).

$$\begin{aligned}\left[\hat{a},\mathrm{e}^{\alpha\,\hat{a}^\dagger}\right] &= \sum_{i=0}^{\infty}\frac{\alpha^i}{i!}\left[\hat{a},(\hat{a}^\dagger)^i\right]\\ &\overset{(3)}{=} \sum_{i=0}^{\infty}\frac{\alpha^i\,i}{i!}(\hat{a}^\dagger)^{i-1}\\ &= \alpha\sum_{i=1}^{\infty}\frac{\alpha^{i-1}}{(i-1)!}(\hat{a}^\dagger)^{i-1}\\ &= \alpha\sum_{j=0}^{\infty}\frac{\alpha^j}{j!}(\hat{a}^\dagger)^j\\ &= \alpha\,\mathrm{e}^{\alpha\,\hat{a}^\dagger}\ .\end{aligned}$$

□

Proof of (6).

$$
\begin{aligned}
\left[\hat{a}^\dagger, \mathrm{e}^{-\alpha^* \hat{a}}\right] &= \sum_{i=0}^{\infty} \frac{(-\alpha^*)^i}{i!} \left[\hat{a}^\dagger, \hat{a}^i\right] \\
&\overset{(4)}{=} \sum_{i=0}^{\infty} \frac{(-\alpha^*)^i\,(-i)}{i!}\, \hat{a}^{i-1} \\
&= \alpha^* \sum_{i=1}^{\infty} \frac{(-\alpha^*)^{i-1}}{(i-1)!}\, \hat{a}^{i-1} \\
&= \alpha^* \sum_{j=0}^{\infty} \frac{(-\alpha^*)^j}{j!}\, \hat{a}^j \\
&= \alpha^*\, \mathrm{e}^{-\alpha^* \hat{a}} \,.
\end{aligned}
$$

□

Proof of (7).

$$
\begin{aligned}
\mathrm{e}^{\mathrm{i}\,k\,\hat{a}^\dagger \hat{a}}\, \hat{a}\, \mathrm{e}^{-\mathrm{i}\,k\,\hat{a}^\dagger \hat{a}} &\overset{(36)}{=} \hat{a} + (-\mathrm{i}\,k)\left[\hat{a}, \hat{a}^\dagger \hat{a}\right] + \frac{(-\mathrm{i}\,k)^2}{2}\left[\left[\hat{a}, \hat{a}^\dagger \hat{a}\right], \hat{a}^\dagger \hat{a}\right] + \cdots \\
&\overset{(2)}{=} \hat{a} + (-\mathrm{i}\,k)\,\hat{a} + \frac{(-\mathrm{i}\,k)^2}{2}\,\hat{a} + \cdots \\
&= \hat{a}\,\mathrm{e}^{-\mathrm{i}\,k} \,.
\end{aligned}
$$

□

Proof of (9). First, we will need the displacement operator in $\hat{x}$ and $\hat{p}$,

$$
\begin{aligned}
\hat{D}[\alpha] &= \mathrm{e}^{\alpha\,\hat{a}^\dagger - \alpha^*\,\hat{a}} \\
&= \mathrm{e}^{\sqrt{\frac{m\,\Omega}{2\,\hbar}}\,(\alpha-\alpha^*)\,\hat{x} - \frac{\mathrm{i}}{\sqrt{2\,\hbar\,m\,\Omega}}\,(\alpha+\alpha^*)\,\hat{p}} \\
&= \mathrm{e}^{-\frac{1}{4}\,(\alpha-\alpha^*)(\alpha+\alpha^*)}\, \mathrm{e}^{\sqrt{\frac{m\,\Omega}{2\,\hbar}}\,(\alpha-\alpha^*)\,\hat{x}}\, \mathrm{e}^{-\frac{\mathrm{i}}{\sqrt{2\,\hbar\,m\,\Omega}}\,(\alpha+\alpha^*)\,\hat{p}} \\
&= \mathrm{e}^{-\mathrm{i}\,\Re[\alpha]\,\Im[\alpha]}\, \mathrm{e}^{\mathrm{i}\sqrt{\frac{2\,m\,\Omega}{\hbar}}\,\Im[\alpha]\,\hat{x}}\, \mathrm{e}^{-\mathrm{i}\sqrt{\frac{2}{\hbar\,m\,\Omega}}\,\Re[\alpha]\,\hat{p}} \,.
\end{aligned}
$$

Now

$$
\begin{aligned}
\langle x \,|\, \alpha \rangle &= \left\langle x \middle| \hat{D}[\alpha] \middle| 0 \right\rangle \\
&= \int_{-\infty}^{\infty} \delta(x - x')\, \mathrm{e}^{-\mathrm{i}\,\Re[\alpha]\,\Im[\alpha]}\, \mathrm{e}^{\mathrm{i}\sqrt{\frac{2\,m\,\Omega}{\hbar}}\,\Im[\alpha]\,x'}\, \mathrm{e}^{-\sqrt{\frac{2\,\hbar}{m\,\Omega}}\,\Re[\alpha]\,\frac{\partial}{\partial x'}}\, \psi_0(x')\,\mathrm{d}x' \\
&= \int_{-\infty}^{\infty} \delta(x - x') \left(\frac{m\,\Omega}{\pi\,\hbar}\right)^{\frac{1}{4}} \mathrm{e}^{\mathrm{i}\,\Im[\alpha]\left(\sqrt{\frac{2\,m\,\Omega}{\hbar}}\,x' - \Re[\alpha]\right) - \left(\sqrt{\frac{m\,\Omega}{2\,\hbar}}\,x' - \Re[\alpha]\right)^2} \mathrm{d}x' \\
&= \left(\frac{m\,\Omega}{\pi\,\hbar}\right)^{\frac{1}{4}} \mathrm{e}^{\mathrm{i}\,\Im[\alpha]\left(\sqrt{\frac{2\,m\,\Omega}{\hbar}}\,x - \Re[\alpha]\right) - \left(\sqrt{\frac{m\,\Omega}{2\,\hbar}}\,x - \Re[\alpha]\right)^2} \,,
\end{aligned}
$$

using $\mathrm{e}^{a\,\frac{\partial}{\partial x}} f(x) = f(x + a)$. □

Proof of (10).

$$
\begin{aligned}
|\psi(t)\rangle &= \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}(t-t_0)}\,|\alpha_0\rangle \\
&= \mathrm{e}^{-\mathrm{i}\,\Omega\,(\hat{a}^\dagger\hat{a}+1/2)(t-t_0)}\,\mathrm{e}^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{\alpha_0{}^n}{\sqrt{n!}}\,|n\rangle \\
&= \mathrm{e}^{-\mathrm{i}\,\Omega\,(t-t_0)/2}\,\mathrm{e}^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{\left(\alpha_0\,\mathrm{e}^{-\mathrm{i}\,\Omega\,(t-t_0)}\right)^n}{\sqrt{n!}}\,|n\rangle \\
&= \mathrm{e}^{-\mathrm{i}\,\Omega\,(t-t_0)/2}\,\left|\alpha_0\,\mathrm{e}^{-\mathrm{i}\,\Omega\,(t-t_0)}\right\rangle\ .
\end{aligned}
$$

□

Proof of (11). Let us denote

$$
\hat{A}(t) = \alpha_0\,\mathrm{e}^{-\mathrm{i}\,\omega_\alpha\,t}\,\hat{a}^\dagger - \alpha_0^*\,\mathrm{e}^{\mathrm{i}\,\omega_\alpha\,t}\,\hat{a} \quad\Rightarrow\quad \hat{D}[\alpha_0\,\mathrm{e}^{-\mathrm{i}\,\omega_\alpha\,t}] = \mathrm{e}^{\hat{A}(t)}\ ,
$$

and compute

$$
\left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{A}(t)\right) = -\,\mathrm{i}\,\omega_\alpha\left(\alpha_0\,\mathrm{e}^{-\mathrm{i}\,\omega_\alpha\,t}\,\hat{a}^\dagger + \alpha_0^*\,\mathrm{e}^{\mathrm{i}\,\omega_\alpha\,t}\,\hat{a}\right)\ ,
$$

$$
\begin{aligned}
\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{A}(t)\right),\hat{A}(t)\right] &= -\,\mathrm{i}\,\omega_\alpha\left(\alpha_0\,\mathrm{e}^{-\mathrm{i}\,\omega_\alpha\,t}\,\alpha_0^*\,\mathrm{e}^{\mathrm{i}\,\omega_\alpha\,t}\,[\hat{a}^\dagger,\hat{a}] - \alpha_0^*\,\mathrm{e}^{\mathrm{i}\,\omega_\alpha\,t}\,\alpha_0\,\mathrm{e}^{-\mathrm{i}\,\omega_\alpha\,t}\,[\hat{a},\hat{a}^\dagger]\right) \\
&= 2\,\mathrm{i}\,\omega_\alpha\,|\alpha_0|^2\ .
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\hat{D}[\alpha_0\,\mathrm{e}^{-\mathrm{i}\,\omega_\alpha\,t}]^\dagger\left(-\mathrm{i}\,\hbar\,\frac{\mathrm{d}}{\mathrm{d}t}\right)\hat{D}[\alpha_0\,\mathrm{e}^{-\mathrm{i}\,\omega_\alpha\,t}] \\
&\overset{(36)}{=} -\,\mathrm{i}\,\hbar\,\frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{i}\,\hbar\left[\frac{\mathrm{d}}{\mathrm{d}t},\hat{A}(t)\right] - \frac{\mathrm{i}\,\hbar}{2}\left[\left[\frac{\mathrm{d}}{\mathrm{d}t},\hat{A}(t)\right],\hat{A}(t)\right] \\
&\qquad - \frac{\mathrm{i}\,\hbar}{6}\left[\left[\left[\frac{\mathrm{d}}{\mathrm{d}t},\hat{A}(t)\right],\hat{A}(t)\right],\hat{A}(t)\right] + \cdots \\
&= -\,\mathrm{i}\,\hbar\,\frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{i}\,\hbar\left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{A}(t)\right) - \hbar\,\omega_\alpha\,|\alpha_0|^2 + 0 \\
&= -\,\mathrm{i}\,\hbar\,\frac{\mathrm{d}}{\mathrm{d}t} - \hbar\,\omega_\alpha\left(\alpha_0\,\mathrm{e}^{-\mathrm{i}\,\omega_\alpha\,t}\,\hat{a}^\dagger + \alpha_0^*\,\mathrm{e}^{\mathrm{i}\,\omega_\alpha\,t}\,\hat{a} + |\alpha_0|^2\right)
\end{aligned}
$$

□

Proof of (12).

$$
\left\langle\zeta\,\middle|\,\hat{x}^2\,\middle|\,\zeta\right\rangle = \frac{\hbar}{2\,m\,\Omega}\,\left\langle 0\,\middle|\,\hat{S}[\zeta]^\dagger\,(\hat{a}\,\hat{a} + \hat{a}^\dagger\,\hat{a}^\dagger + \hat{a}^\dagger\,\hat{a} + \hat{a}\,\hat{a}^\dagger)\,\hat{S}[\zeta]\,\middle|\,0\right\rangle\ .
$$

Let

$$\hat{G} = \frac{\zeta^* \, \hat{a}\,\hat{a} - \zeta\,\hat{a}^\dagger\,\hat{a}^\dagger}{2} \quad : \quad \hat{S}[\zeta] = \mathrm{e}^{\hat{G}} \, ,$$

$$\hat{A} = \hat{a}^\dagger\,\hat{a} + \hat{a}\,\hat{a}^\dagger \, ,$$

$$\hat{B} = \zeta^*\,\hat{a}\,\hat{a} + \zeta\,\hat{a}^\dagger\,\hat{a}^\dagger \, ,$$

$$R = \frac{\zeta + \zeta^*}{2} \, ,$$

and we will need the following commutation relations

$$\left[\hat{a}\,\hat{a}, \hat{G}\right] = -\zeta\,\hat{A} \, , \qquad \left[\hat{a}^\dagger\,\hat{a}, \hat{G}\right] = -\hat{B} \, ,$$

$$\left[\hat{a}^\dagger\,\hat{a}^\dagger, \hat{G}\right] = -\zeta^*\,\hat{A} \, , \qquad \left[\hat{a}\,\hat{a}^\dagger, \hat{G}\right] = -\hat{B} \, ,$$

$$\left[\hat{A}, \hat{G}\right] = -2\,\hat{B} \, , \qquad \left[\hat{B}, \hat{G}\right] = -2\,|\zeta|^2\,\hat{A} \, .$$

To use the Baker-Campbell-Hausdorff equation (36), compute

$$\left[\hat{a}\,\hat{a} + \hat{a}^\dagger\,\hat{a}^\dagger + \hat{A}, \hat{G}\right] = -2\,R\,\hat{A} - 2\,\hat{B} = \hat{C}_1 \, ,$$

$$\left[\hat{C}_1, \hat{G}\right] = 4\,R\,\hat{B} + 4\,|\zeta|^2\,\hat{A} = \hat{C}_2 \, ,$$

$$\left[\hat{C}_2, \hat{G}\right] = -8\,R\,|\zeta|^2\,\hat{A} - 8\,|\zeta|^2\,\hat{B} = \hat{C}_3 \, ,$$

$$\left[\hat{C}_3, \hat{G}\right] = 16\,R\,|\zeta|^2\,\hat{B} + 16\,|\zeta|^4\,\hat{A} = \hat{C}_4 \, ,$$

$$\cdots \, ,$$

then

$$\begin{aligned}\hat{S}[\zeta]^\dagger\left(\hat{a}\,\hat{a} + \hat{a}^\dagger\,\hat{a}^\dagger + \hat{A}\right)\hat{S}[\zeta] = \hat{a}\,\hat{a} + \hat{a}^\dagger\,\hat{a}^\dagger + \hat{A} \\ -2\left(R\,\hat{A} + \hat{B}\right) + \frac{2^2}{2!}\left(R\,\hat{B} + |\zeta|^2\,\hat{A}\right) \\ -\frac{2^3}{3!}\left(R\,|\zeta|^2\,\hat{A} + |\zeta|^2\,\hat{B}\right) + \frac{2^4}{4!}\left(R\,|\zeta|^2\,\hat{B} + |\zeta|^4\,\hat{A}\right) + \cdots \, .\end{aligned}$$

When taking the vacuum expectation, only terms of $\hat{a}\,\hat{a}^\dagger \in \hat{A}$ survive with

$$\left\langle 0 \middle| \hat{A} \middle| 0 \right\rangle = 0 + \left\langle 0 \middle| \hat{a}\,\hat{a}^\dagger \middle| 0 \right\rangle = 1 \, .$$

Thus,

$$\langle \zeta \,|\, \hat{x}^2 \,|\, \zeta \rangle = \frac{\hbar}{2\,m\,\Omega} \left(1 - 2\,R + \frac{2^2}{2!}\,|\zeta|^2 - \frac{2^3}{3!}\,R\,|\zeta|^2 + \frac{2^4}{4!}\,|\zeta|^4 + \cdots\right)$$

$$= \frac{\hbar}{2\,m\,\Omega} \left(1 + \frac{(2\,|\zeta|)^2}{2!} + \frac{(2\,|\zeta|)^4}{4!} + \frac{(2\,|\zeta|)^6}{6!} + \cdots\right) - \frac{\hbar}{2\,m\,\Omega}\,\frac{R}{|\zeta|} \left(2\,|\zeta| + \frac{(2\,|\zeta|)^3}{3!} + \frac{(2\,|\zeta|)^5}{5!} + \cdots\right)$$

$$= \frac{\hbar}{2\,m\,\Omega} \left(\cosh[2\,|\zeta|] - \Re\left[\sqrt{\zeta/\zeta^*}\right] \sinh[2\,|\zeta|]\right).$$

□

Proof of (13). Using (3) and (4),

$$\hat{S}[\zeta]^\dagger\,\hat{a}\,\hat{S}[\zeta] \overset{(36)}{=} \hat{a} + (-\zeta\,\hat{a}^\dagger) + \frac{1}{2!}\left(|\zeta|^2\,\hat{a}\right) + \frac{1}{3!}\left(-|\zeta|^2\,\zeta\,\hat{a}^\dagger\right) + \frac{1}{4!}\left(|\zeta|^4\,\hat{a}\right) + \cdots$$

$$= \hat{a}\,\cosh\big[|\zeta|\big] - \hat{a}^\dagger\,\frac{|\zeta|}{\zeta^*}\,\sinh\big[|\zeta|\big].$$

□

Proof of (15).

$$\left[\hat{x}, \hat{p}^2\right] = 2\,\mathrm{i}\,\hbar\,\hat{p}\,, \qquad\qquad \left[\hat{p}, \hat{x}^2\right] = -2\,\mathrm{i}\,\hbar\,\hat{x}\,, \tag{44}$$

$$\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\hat{H}\,t}\,\hat{x}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}\,t} \overset{(36)}{=} \hat{x} + \left[\hat{x}, \frac{-\mathrm{i}\,t}{\hbar}\,\frac{\hat{p}^2}{2\,m}\right] + \cdots$$

$$\overset{(44)}{=} \hat{x} + \frac{t}{m}\,\hat{p} + \frac{1}{2}\left[\frac{t}{m}\,\hat{p}, \frac{-\mathrm{i}\,t}{\hbar}\,\frac{m\,\Omega^2\,\hat{x}^2}{2}\right] + \cdots$$

$$\overset{(44)}{=} \hat{x} + \frac{t}{m}\,\hat{p} + \frac{1}{2}\left(-\Omega^2\,t^2\,\hat{x}\right) + \frac{1}{6}\left(-\frac{\Omega^2\,t^3}{2}\,\hat{p}\right) + \frac{1}{24}\left(\Omega^4\,t^4\,\hat{x}\right) + \cdots$$

$$= \hat{x}\,\cos[\Omega\,t] + \frac{\hat{p}}{m\,\Omega}\,\sin[\Omega\,t]\,.$$

□

Proof of (16).

$$\left\langle \cos[\Omega\, t]^2 \right\rangle_t = \lim_{T\to\infty} \frac{1}{T} \int_0^T \cos[\Omega\, t]^2 \,\mathrm{d}t = \frac{1}{2} \,,$$

$$\left\langle \sin[\Omega\, t]^2 \right\rangle_t = \lim_{T\to\infty} \frac{1}{T} \int_0^T \sin[\Omega\, t]^2 \,\mathrm{d}t = \frac{1}{2} \,,$$

$$\langle \sin[\Omega\, t]\, \cos[\Omega\, t] \rangle_t = \lim_{T\to\infty} \frac{1}{T} \int_0^T \sin[\Omega\, t]\, \cos[\Omega\, t] \,\mathrm{d}t = 0 \,.$$

$$\begin{aligned}
\left\langle \hat{x}(t)^\dagger\, \hat{x}(t+\tau) \right\rangle_t &= \left\langle \hat{x}^2 \cos[\Omega\, t]\, \cos[\Omega\,(t+\tau)] \right\rangle_t + \left\langle \frac{\hat{x}\,\hat{p}}{m\,\Omega} \cos[\Omega\, t]\, \sin[\Omega\,(t+\tau)] \right\rangle_t \\
&\quad + \left\langle \frac{\hat{p}\,\hat{x}}{m\,\Omega} \sin[\Omega\, t]\, \cos[\Omega\,(t+\tau)] \right\rangle_t + \left\langle \frac{\hat{p}^2}{m^2\,\Omega^2} \sin[\Omega\, t]\, \sin[\Omega\,(t+\tau)] \right\rangle_t \\
&= \left\langle \hat{x}^2 \right\rangle \left\langle \cos[\Omega\, t]^2 \right\rangle_t \cos[\Omega\,\tau] - \left\langle \hat{x}^2 \right\rangle \langle \cos[\Omega\, t]\, \sin[\Omega\, t] \rangle_t \sin[\Omega\,\tau] \\
&\quad + \left\langle \frac{\hat{x}\,\hat{p}}{m\,\Omega} \right\rangle \langle \cos[\Omega\, t]\, \sin[\Omega\, t] \rangle_t \cos[\Omega\,\tau] + \left\langle \frac{\hat{x}\,\hat{p}}{m\,\Omega} \right\rangle \left\langle \cos[\Omega\, t]^2 \right\rangle_t \sin[\Omega\,\tau] \\
&\quad + \left\langle \frac{\hat{p}\,\hat{x}}{m\,\Omega} \right\rangle \langle \sin[\Omega\, t]\, \cos[\Omega\, t] \rangle_t \cos[\Omega\,\tau] - \left\langle \frac{\hat{p}\,\hat{x}}{m\,\Omega} \right\rangle \left\langle \sin[\Omega\, t]^2 \right\rangle_t \sin[\Omega\,\tau] \\
&\quad + \left\langle \frac{\hat{p}^2}{m^2\,\Omega^2} \right\rangle \left\langle \sin[\Omega\, t]^2 \right\rangle_t \cos[\Omega\,\tau] + \left\langle \frac{\hat{p}^2}{m^2\,\Omega^2} \right\rangle \langle \sin[\Omega\, t]\, \cos[\Omega\, t] \rangle_t \sin[\Omega\,\tau] \\
&= \frac{1}{2} \left\langle \hat{x}^2 + \frac{\hat{p}^2}{m^2\,\Omega^2} \right\rangle \cos[\Omega\,\tau] + \frac{1}{2} \left\langle \frac{\hat{x}\,\hat{p} - \hat{p}\,\hat{x}}{m\,\Omega} \right\rangle \sin[\Omega\,\tau] \\
&= \frac{\langle \hat{H} \rangle}{m\,\Omega^2} \cos[\Omega\,\tau] + \frac{\mathrm{i}\,\hbar}{2\,m\,\Omega} \sin[\Omega\,\tau] \\
&= \frac{\hbar}{m\,\Omega} \left(\langle \hat{a}^\dagger\, \hat{a} \rangle \cos[\Omega\,\tau] + \frac{1}{2}\, \mathrm{e}^{\mathrm{i}\,\Omega\,\tau} \right) .
\end{aligned}$$

Alternatively,

$$\hat{x}(t) \overset{(15)}{=} \sqrt{\frac{\hbar}{2\,m\,\Omega}} \left(\hat{a}^\dagger\, \mathrm{e}^{\mathrm{i}\,\Omega\,t} + \hat{a}\, \mathrm{e}^{-\mathrm{i}\,\Omega\,t} \right) .$$

$$
\begin{aligned}
\left\langle \hat{x}(t)^\dagger\, \hat{x}(t+\tau)\right\rangle_t &= \frac{\hbar}{2\,m\,\Omega}\left(\left\langle \hat{a}^\dagger\, \hat{a}\right\rangle \mathrm{e}^{-\mathrm{i}\,\Omega\,\tau} + \left\langle \hat{a}\,\hat{a}^\dagger\right\rangle \mathrm{e}^{\mathrm{i}\,\Omega\,\tau}\right) \\
&= \frac{\hbar}{2\,m\,\Omega}\left(\cos[\Omega\,t]\left(\left\langle \hat{a}^\dagger\, \hat{a}\right\rangle + \left\langle \hat{a}\,\hat{a}^\dagger\right\rangle\right) + \mathrm{i}\,\sin[\Omega\,t]\left(\left\langle \hat{a}\,\hat{a}^\dagger\right\rangle - \left\langle \hat{a}^\dagger\, \hat{a}\right\rangle\right)\right) \\
&= \frac{\hbar}{2\,m\,\Omega}\left(\cos[\Omega\,t]\left(2\,\left\langle \hat{a}^\dagger\, \hat{a}\right\rangle + 1\right) + \mathrm{i}\,\sin[\Omega\,t]\right) \\
&= \frac{\hbar}{m\,\Omega}\left(\left\langle \hat{a}^\dagger\, \hat{a}\right\rangle \cos[\Omega\,\tau] + \frac{1}{2}\,\mathrm{e}^{\mathrm{i}\,\Omega\,\tau}\right).
\end{aligned}
$$

□

Proof of (17). For $\hat{H}_{\text{sys}} = \hbar\,\Omega\left(\hat{a}^\dagger\,\hat{a} + \frac{1}{2}\right)$. In the interaction-picture with $\hat{U}(t) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(\hat{H}_{\text{sys}}+\hat{H}_{\text{bath}})\,t}$,

$$
\begin{aligned}
\mathcal{U}\left[\hat{H}_{\text{int}}\right] &= \hat{U}(t)^\dagger\,\hat{H}_{\text{int}}\,\hat{U}(t) \\
&= \sum_q \hbar\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\hat{H}_{\text{sys}}\,t}\,\left(\kappa_q\,\hat{a}^\dagger + \kappa_q^*\,\hat{a}\right)\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}_{\text{sys}}\,t}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\hat{H}_{\text{bath}}\,t}\,\left(\hat{b}_q^\dagger + \hat{b}_q\right)\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}_{\text{bath}}\,t} \\
&\overset{(7)}{=} \sum_q \hbar\left(\kappa_q\,\hat{a}^\dagger\,\mathrm{e}^{\mathrm{i}\,\Omega\,t} + \kappa_q^*\,\hat{a}\,\mathrm{e}^{-\mathrm{i}\,\Omega\,t}\right)\left(\hat{b}_q^\dagger\,\mathrm{e}^{\mathrm{i}\,\omega_q\,t} + \hat{b}_q\,\mathrm{e}^{-\mathrm{i}\,\omega_q\,t}\right) \\
&= \sum_q \hbar\Big(\kappa_q^*\,\hat{a}\,\hat{b}_q\,\mathrm{e}^{-\mathrm{i}\,(\Omega+\omega_q)t} + \kappa_q^*\,\hat{a}\,\hat{b}_q^\dagger\,\mathrm{e}^{-\mathrm{i}\,(\Omega-\omega_q)t} \\
&\qquad + \kappa_q\,\hat{a}^\dagger\,\hat{b}_q\,\mathrm{e}^{\mathrm{i}\,(\Omega-\omega_q)t} + \kappa_q\,\hat{a}^\dagger\,\hat{b}_q^\dagger\,\mathrm{e}^{\mathrm{i}\,(\Omega+\omega_q)t}\Big)\,.
\end{aligned}
$$

To make the rotating-wave approximation, average over short timescales $T \sim \frac{1}{\Omega+\omega_q} < \frac{1}{\Omega-\omega_q}$,

$$
\mathcal{U}\left[\hat{H}_{\text{int}}\right] \approx \sum_q \hbar\left(\kappa_q^*\,\hat{a}\,\hat{b}_q^\dagger\,\mathrm{e}^{-\mathrm{i}\,(\Omega-\omega_q)t} + \kappa_q\,\hat{a}^\dagger\,\hat{b}_q\,\mathrm{e}^{\mathrm{i}\,(\Omega-\omega_q)t}\right)\,.
$$

□

Proof of (19).

$$
\begin{aligned}
\left[\hat{b}_{\text{in}}(t), \hat{b}_{\text{in}}(t')^\dagger\right] &= \frac{1}{2\,\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\,\omega\,(t-t_0)}\,\mathrm{e}^{\mathrm{i}\,\omega'\,(t'-t_0)}\,\left[\hat{b}(\omega,t_0), \hat{b}(\omega',t_0)^\dagger\right]\,\mathrm{d}\omega\,\mathrm{d}\omega' \\
&= \frac{1}{2\,\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\,\omega\,(t-t_0)+\mathrm{i}\,\omega'\,(t'-t_0)}\,\delta(\omega-\omega')\,\mathrm{d}\omega\,\mathrm{d}\omega' \\
&= \frac{1}{2\,\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\,\omega\,(t-t')}\,\mathrm{d}\omega \\
&\overset{(40)}{=} \delta(t-t')\,.
\end{aligned}
$$

□

Proof of (22).

$$
\begin{aligned}
\left[\hat{b}_{\text{out}}(t), \hat{b}_{\text{out}}(t')^{\dagger}\right] &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\,\omega\,(t_1-t)}\,\mathrm{e}^{-\mathrm{i}\,\omega'\,(t_1-t')}\,\left[\hat{b}(\omega,t_1),\hat{b}(\omega',t_1)^{\dagger}\right]\mathrm{d}\omega\,\mathrm{d}\omega' \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\,\omega\,(t_1-t)-\mathrm{i}\,\omega'\,(t_1-t')}\,\delta(\omega-\omega')\,\mathrm{d}\omega\,\mathrm{d}\omega' \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\,\omega\,(t-t')}\,\mathrm{d}\omega \\
&\overset{(40)}{=} \delta(t-t')\,.
\end{aligned}
$$

□

Proof of (27).

$$
\begin{aligned}
\mathbb{P}_{n\to n-1} &\approx n\,|\gamma|\int_{t_0}^{t}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\,\Omega\,\tau}\,\left\langle \hat{b}_{\text{in}}(t')\,\hat{b}_{\text{in}}^{\dagger}(t'+\tau)\right\rangle\mathrm{d}\tau\,\mathrm{d}t' \\
&= n\,|\gamma|\int_{t_0}^{t}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\,\Omega\,\tau}\,\left[\hat{b}_{\text{in}}(t'),\hat{b}_{\text{in}}^{\dagger}(t'+\tau)\right]\mathrm{d}\tau\,\mathrm{d}t' \\
&\quad + n\,|\gamma|\int_{t_0}^{t}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\,\Omega\,\tau}\,\left\langle \hat{b}_{\text{in}}^{\dagger}(t'+\tau)\,\hat{b}_{\text{in}}(t')\right\rangle\mathrm{d}\tau\,\mathrm{d}t' \\
&= n\,|\gamma|\int_{t_0}^{t}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\,\Omega\,\tau}\,\delta[\tau]\,\mathrm{d}\tau\,\mathrm{d}t' \\
&\quad + n\,|\gamma|\int_{t_0}^{t}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\,\Omega\,\tau}\,\left\langle \hat{b}_{\text{in}}^{\dagger}(t'-\tau)\,\hat{b}_{\text{in}}(t')\right\rangle\mathrm{d}\tau\,\mathrm{d}t' \quad : \quad \tau \longmapsto -\tau \\
&= n\,|\gamma|\,(t-t_0)\left(S_{\hat{b}_{\text{in}}\hat{b}_{\text{in}}}(\Omega)+1\right).
\end{aligned}
$$

□

Proof of (26).

$$
\begin{aligned}
\underset{\text{Fock}}{R_{\hat{b}_{\text{in}}\hat{b}_{\text{in}}}}(\tau) &= \left\langle \bigotimes_{\omega} n(\omega,t_0)\middle|\,\hat{b}_{\text{in}}^{\dagger}(t)\,\hat{b}_{\text{in}}(t+\tau)\,\middle|\bigotimes_{\omega} n(\omega,t_0)\right\rangle \\
&= \frac{1}{2\pi}\left\langle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\,\omega''(t-t_0)}\,\mathrm{e}^{-\mathrm{i}\,\omega'\,(t+\tau-t_0)}\,\hat{b}^{\dagger}(\omega'',t_0)\,\hat{b}(\omega',t_0)\,\mathrm{d}\omega''\,\mathrm{d}\omega'\right\rangle \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\langle \mathrm{e}^{-\mathrm{i}\,(\omega'-\omega'')\,(t-t_0)}\right\rangle\mathrm{e}^{-\mathrm{i}\,\omega'\,\tau}\,\delta(\omega''-\omega')\,n(\omega',t_0)\,\mathrm{d}\omega''\,\mathrm{d}\omega' \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\,\omega'\,\tau}\,n(\omega',t_0)\,\mathrm{d}\omega'\,.
\end{aligned}
$$

$$
\begin{aligned}
S_{\underset{\text{Fock}}{\hat b_{\text{in}} \hat b_{\text{in}}}}(\omega) &= \int_{-\infty}^{\infty} R_{\underset{\text{Fock}}{\hat b_{\text{in}} \hat b_{\text{in}}}}(\tau)\, \mathrm{e}^{\mathrm{i}\,\omega\,\tau}\, \mathrm{d}\tau \\
&= \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\,(\omega'-\omega)\,\tau}\, n(\omega', t_0)\, \mathrm{d}\tau\, \mathrm{d}\omega' \\
&= \int_{-\infty}^{\infty} \delta(\omega - \omega')\, n(\omega', t_0)\, \mathrm{d}\omega' \\
&= n(\omega, t_0)\ .
\end{aligned}
$$

$$
\begin{aligned}
R_{\underset{\text{Fock}}{\hat b^\dagger_{\text{in}} \hat b^\dagger_{\text{in}}}}(\tau) &= \left\langle \bigotimes_{\omega} n(\omega, t_0) \middle| \hat b_{\text{in}}(t)\, \hat b^\dagger_{\text{in}}(t+\tau) \middle| \bigotimes_{\omega} n(\omega, t_0) \right\rangle \\
&= \frac{1}{2\,\pi} \left\langle \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\,\omega''(t-t_0)}\, \mathrm{e}^{\mathrm{i}\,\omega'\,(t+\tau-t_0)}\, \hat b(\omega'', t_0)\, \hat b^\dagger(\omega', t_0)\, \mathrm{d}\omega''\, \mathrm{d}\omega' \right\rangle \\
&= \frac{1}{2\,\pi} \left\langle \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\,\omega''(t-t_0)}\, \mathrm{e}^{\mathrm{i}\,\omega'\,(t+\tau-t_0)} \right. \\
&\qquad \left. \times \Big(\big[\hat b(\omega'', t_0), \hat b^\dagger(\omega', t_0)\big] + \hat b^\dagger(\omega'', t_0)\, \hat b(\omega', t_0) \Big)\, \mathrm{d}\omega''\, \mathrm{d}\omega' \right\rangle \\
&= \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\langle \mathrm{e}^{\mathrm{i}\,(\omega'-\omega'')\,(t-t_0)} \right\rangle \mathrm{e}^{\mathrm{i}\,\omega'\,\tau}\, \delta(\omega'' - \omega') \big(n(\omega', t_0) + 1 \big)\, \mathrm{d}\omega''\, \mathrm{d}\omega' \\
&= \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\,\omega'\,\tau} \big(n(\omega', t_0) + 1 \big)\, \mathrm{d}\omega'\ .
\end{aligned}
$$

$$
\begin{aligned}
S_{\underset{\text{Fock}}{\hat b^\dagger_{\text{in}} \hat b^\dagger_{\text{in}}}}(\omega) &= \int_{-\infty}^{\infty} R_{\underset{\text{Fock}}{\hat b^\dagger_{\text{in}} \hat b^\dagger_{\text{in}}}}(\tau)\, \mathrm{e}^{\mathrm{i}\,\omega\,\tau}\, \mathrm{d}\tau \\
&= \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\,(\omega'+\omega)\,\tau} \big(n(\omega', t_0) + 1 \big)\, \mathrm{d}\tau\, \mathrm{d}\omega' \\
&= \int_{-\infty}^{\infty} \delta(\omega + \omega') \big(n(\omega', t_0) + 1 \big)\, \mathrm{d}\omega' \\
&= n(-\omega, t_0) + 1\ .
\end{aligned}
$$

$$
\begin{aligned}
R_{\underset{\text{Fock}}{\hat b_{\text{out}} \hat b_{\text{out}}}}(\tau) &= \left\langle \bigotimes_{\omega} n(\omega, t_0) \middle| \hat b^\dagger_{\text{out}}(t)\, \hat b_{\text{out}}(t+\tau) \middle| \bigotimes_{\omega} n(\omega, t_0) \right\rangle \\
&= \frac{1}{2\,\pi} \left\langle \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\,\omega''(t_1-t)}\, \mathrm{e}^{\mathrm{i}\,\omega'\,(t_1-t-\tau)}\, \hat b^\dagger(\omega'', t_1)\, \hat b(\omega', t_1)\, \mathrm{d}\omega''\, \mathrm{d}\omega' \right\rangle \\
&= \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\langle \mathrm{e}^{\mathrm{i}\,(\omega'-\omega'')\,(t_1-t)} \right\rangle \mathrm{e}^{-\mathrm{i}\,\omega'\,\tau}\, \delta(\omega'' - \omega')\, n(\omega', t_1)\, \mathrm{d}\omega''\, \mathrm{d}\omega' \\
&= \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\,\omega'\,\tau}\, n(\omega', t_1)\, \mathrm{d}\omega'\ .
\end{aligned}
$$

$$
\begin{aligned}
R_{\hat{b}^\dagger_{\text{out}}\hat{b}^\dagger_{\text{out}}}^{\text{Fock}}(\tau) &= \left\langle \bigotimes_\omega n(\omega, t_0) \middle| \hat{b}_{\text{out}}(t)\, \hat{b}^\dagger_{\text{out}}(t+\tau) \middle| \bigotimes_\omega n(\omega, t_0) \right\rangle \\
&= \frac{1}{2\pi} \left\langle \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\omega''(t_1-t)}\, e^{-i\omega'(t_1-t-\tau)}\, \hat{b}(\omega'', t_1)\, \hat{b}^\dagger(\omega', t_1)\, d\omega''\, d\omega' \right\rangle \\
&= \frac{1}{2\pi} \left\langle \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\omega''(t_1-t)}\, e^{-i\omega'(t_1-t-\tau)} \right. \\
&\qquad\qquad \left. \times \Big(\big[\hat{b}(\omega'', t_1), \hat{b}^\dagger(\omega', t_1)\big] + \hat{b}^\dagger(\omega'', t_1)\, \hat{b}(\omega', t_1) \Big)\, d\omega''\, d\omega' \right\rangle \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\langle e^{-i(\omega'-\omega'')(t_1-t)} \right\rangle e^{i\omega'\tau}\, \delta(\omega''-\omega') \big(n(\omega', t_1) + 1\big)\, d\omega''\, d\omega' \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega'\tau} \big(n(\omega', t_1) + 1\big)\, d\omega' \,.
\end{aligned}
$$

□

Proof of (25). The time-derivative has Fourier transform,

$$
\begin{aligned}
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} \left(\frac{d}{dt}\hat{a}(t)\right) dt &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d}{dt}\Big(e^{i\omega t}\, \hat{a}(t)\Big) dt - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} i\,\omega\, e^{i\omega t}\, \hat{a}(t)\, dt \\
&= \underbrace{\frac{1}{\sqrt{2\pi}}\, e^{i\omega t}\, \hat{a}(t)\Big|_{t=-\infty}^{t=\infty}}_{0} - i\,\omega\, \hat{a}(\omega) \,,
\end{aligned}
$$

where the marked term is zero when our oscillator has a finite bandwidth, thus $\hat{a}(\omega)$ is integrable, and by the Riemann-Lebesgue lemma $\hat{a}(t \to \pm\infty) \to 0$.

For the input operator,

$$
\begin{aligned}
\hat{b}_{\text{in}}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t}\, \hat{b}_{\text{in}}(t)\, dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\omega t}\, e^{-i\omega'(t-t_0)}\, \hat{b}(\omega', t_0)\, d\omega'\, dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\omega-\omega')t}\, dt\, e^{i\omega' t_0}\, \hat{b}(\omega', t_0)\, d\omega' \\
&= \int_{-\infty}^{\infty} \delta(\omega-\omega')\, e^{i\omega' t_0}\, \hat{b}(\omega', t_0)\, d\omega' \\
&= e^{i\omega t_0}\, \hat{b}(\omega, t_0) \,.
\end{aligned}
$$

For the output operator,

$$\begin{aligned}
\hat{b}_{\text{out}}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\,\omega\,t}\, \hat{b}_{\text{out}}(t)\, dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\,\omega\,t}\, e^{i\,\omega'\,(t_1 - t)}\, \hat{b}(\omega', t_1)\, d\omega'\, dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\,(\omega-\omega')\,t}\, dt\, e^{i\,\omega'\,t_1}\, \hat{b}(\omega', t_1)\, d\omega' \\
&= \int_{-\infty}^{\infty} \delta(\omega - \omega')\, e^{i\,\omega'\,t_1}\, \hat{b}(\omega', t_1)\, d\omega' \\
&= e^{i\,\omega\,t_1}\, \hat{b}(\omega, t_1)\ .
\end{aligned}$$

□

Proof of (28).

$$\begin{aligned}
S_{\hat{b}^\dagger_{\text{in}}\hat{b}^\dagger_{\text{in}}}(-\omega) =\ & \lim_{T\to\infty} \left\langle \hat{b}_{\text{in}\,T}(\omega)\, \hat{b}^\dagger_{\text{in}\,T}(-\omega) \right\rangle \\
=\ & \lim_{T\to\infty} \frac{1}{T} \int_0^T \int_0^T \left\langle \hat{b}_{\text{in}}(t')\, \hat{b}^\dagger_{\text{in}}(t'') \right\rangle e^{-i\,\omega(t''-t')}\, dt'\, dt'' \\
=\ & \lim_{T\to\infty} \frac{1}{T} \int_0^T \int_0^T \left\langle \left[\hat{b}_{\text{in}}(t'), \hat{b}^\dagger_{\text{in}}(t'')\right] \right\rangle e^{-i\,\omega(t''-t')}\, dt'\, dt'' \\
& + \lim_{T\to\infty} \frac{1}{T} \int_0^T \int_0^T \left\langle \hat{b}^\dagger_{\text{in}}(t')\, \hat{b}_{\text{in}}(t'') \right\rangle e^{-i\,\omega(t'-t'')}\, dt'\, dt'' \\
=\ & \lim_{T\to\infty} \frac{1}{T} \int_0^T \int_0^T \delta(t' - t'')\, e^{-i\,\omega(t''-t')}\, dt'\, dt'' + \lim_{T\to\infty} \left\langle \hat{b}^\dagger_{\text{in}\,T}(-\omega)\, \hat{b}_{\text{in}\,T}(\omega) \right\rangle \\
=\ & \lim_{T\to\infty} \frac{1}{T} \int_0^T dt' + S_{\hat{b}_{\text{in}}\hat{b}_{\text{in}}}(\omega) \\
=\ & 1 + S_{\hat{b}_{\text{in}}\hat{b}_{\text{in}}}(\omega)\ .
\end{aligned}$$

□

Proof of (29).

$$S_{(\hat{a}^\dagger+\hat{a})(\hat{a}^\dagger+\hat{a})}(\omega) = \lim_{T\to\infty}\Big(\left\langle \hat{a}_T(-\omega)\,\hat{a}_T^\dagger(\omega)\right\rangle + \left\langle \hat{a}_T^\dagger(-\omega)\,\hat{a}_T(\omega)\right\rangle + \langle \hat{a}_T(-\omega)\,\hat{a}_T(\omega)\rangle + \left\langle \hat{a}_T^\dagger(-\omega)\,\hat{a}_T^\dagger(\omega)\right\rangle\Big)$$

$$= \frac{|\gamma|\lim_{T\to\infty}\left\langle \hat{b}_{\mathrm{in}\,T}(-\omega)\,\hat{b}_{\mathrm{in}\,T}^\dagger(\omega)\right\rangle}{(\omega+\Omega)^2+\frac{|\gamma|^2}{4}} + \frac{|\gamma|\lim_{T\to\infty}\left\langle \hat{b}_{\mathrm{in}\,T}^\dagger(-\omega)\,\hat{b}_{\mathrm{in}\,T}(\omega)\right\rangle}{(\omega-\Omega)^2+\frac{|\gamma|^2}{4}}
+ \frac{\gamma\lim_{T\to\infty}\langle \hat{b}_{\mathrm{in}\,T}(-\omega)\,\hat{b}_{\mathrm{in}\,T}(\omega)\rangle}{\left(\mathrm{i}\,(-\omega-\Omega)-\frac{|\gamma|}{2}\right)\left(\mathrm{i}\,(\omega-\Omega)-\frac{|\gamma|}{2}\right)}
+ \frac{\gamma^*\lim_{T\to\infty}\left\langle \hat{b}_{\mathrm{in}\,T}^\dagger(-\omega)\,\hat{b}_{\mathrm{in}\,T}^\dagger(\omega)\right\rangle}{\left(-\mathrm{i}\,(-\omega-\Omega)-\frac{|\gamma|}{2}\right)\left(-\mathrm{i}\,(\omega-\Omega)-\frac{|\gamma|}{2}\right)}$$

$$= \frac{|\gamma|\,S_{\hat{b}_{\mathrm{in}}^\dagger\hat{b}_{\mathrm{in}}^\dagger}(\omega)}{(\omega+\Omega)^2+\frac{|\gamma|^2}{4}} + \frac{|\gamma|\,S_{\hat{b}_{\mathrm{in}}\hat{b}_{\mathrm{in}}}(\omega)}{(\omega-\Omega)^2+\frac{|\gamma|^2}{4}}$$

$$= \frac{|\gamma|\,\left(S_{\hat{b}_{\mathrm{in}}\hat{b}_{\mathrm{in}}}(-\omega)+1\right)}{(\omega+\Omega)^2+\frac{|\gamma|^2}{4}} + \frac{|\gamma|\,S_{\hat{b}_{\mathrm{in}}\hat{b}_{\mathrm{in}}}(\omega)}{(\omega-\Omega)^2+\frac{|\gamma|^2}{4}}\,.$$

Thus

$$\bar{S}_{(\hat{a}^\dagger+\hat{a})(\hat{a}^\dagger+\hat{a})}(\omega\approx\Omega) = \frac{|\gamma|\,\left(\bar{n}+\frac{1}{2}\right)}{(\omega-\Omega)^2+\frac{|\gamma|^2}{4}}\,,$$

and $\bar{S}_{\hat{x}\hat{x}}(\omega) = \frac{\hbar}{2\,m\,\Omega}\,\hat{S}_{(\hat{a}^\dagger+\hat{a})(\hat{a}^\dagger+\hat{a})}(\omega)$.

□

Proof of (32). First, we will need

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left|\alpha_0\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t}\right\rangle &\overset{(8)}{=} \frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{-\left|\alpha_0\right|^2/2}\sum_{n=0}^{\infty}\frac{{\alpha_0}^n\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t\,n}}{\sqrt{n!}}\;|n\rangle \\
&= \mathrm{e}^{-\left|\alpha_0\right|^2/2}\sum_{n=0}^{\infty}(-\mathrm{i}\,\omega_\mathrm{L}\,n)\,\frac{{\alpha_0}^n\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t\,n}}{\sqrt{n!}}\;|n\rangle \\
&= -\mathrm{i}\,\omega_\mathrm{L}\,\mathrm{e}^{-\left|\alpha_0\right|^2/2}\sum_{n=1}^{\infty}{\alpha_0}^n\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t\,n}\,\frac{n}{\sqrt{n!}}\;|n\rangle \\
&= -\mathrm{i}\,\omega_\mathrm{L}\,\mathrm{e}^{-\left|\alpha_0\right|^2/2}\sum_{m=0}^{\infty}{\alpha_0}^{(m+1)}\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t\,(m+1)}\,\frac{m+1}{\sqrt{(m+1)!}}\;|m+1\rangle \\
&= -\mathrm{i}\,\omega_\mathrm{L}\,\alpha_0\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t}\,\mathrm{e}^{-\left|\alpha_0\right|^2/2}\sum_{m=0}^{\infty}{\alpha_0}^m\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t\,m}\,\frac{\sqrt{(m+1)!}}{m!}\;|m+1\rangle \\
&= -\mathrm{i}\,\omega_\mathrm{L}\,\alpha_0\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t}\,\mathrm{e}^{-\left|\alpha_0\right|^2/2}\sum_{m=0}^{\infty}\frac{{\alpha_0}^m\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t\,m}}{\sqrt{m!}}\;\hat{a}^\dagger\,|m\rangle \\
&= -\mathrm{i}\,\omega_\mathrm{L}\,\alpha_0\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t}\,\hat{a}^\dagger\left|\alpha_0\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t}\right\rangle\;.
\end{aligned}$$

Then,

$$\begin{aligned}
\left(\hat{H}-\mathrm{i}\,\hbar\,\frac{\mathrm{d}}{\mathrm{d}t}\right)\left|\alpha_0\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t}\right\rangle &= \Bigg(\hbar\,\Omega\,\hat{a}^\dagger\,\hat{a}+\frac{\hbar\,\Omega}{2}+\hbar\,L\,\hat{a}^\dagger\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t} \\
&\qquad +\hbar\,L^*\,\hat{a}\,\mathrm{e}^{\mathrm{i}\,\omega_\mathrm{L}\,t}-\mathrm{i}\,\hbar\,\frac{\mathrm{d}}{\mathrm{d}t}\Bigg)\left|\alpha_0\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t}\right\rangle \\
&= \Bigg(\hbar\,\Omega\,\alpha_0\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t}\,\hat{a}^\dagger+\frac{\hbar\,\Omega}{2}+\hbar\,L\,\hat{a}^\dagger\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t} \\
&\qquad +\hbar\,L^*\,\alpha_0\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t}\,\mathrm{e}^{\mathrm{i}\,\omega_\mathrm{L}\,t}-\hbar\,\omega_\mathrm{L}\,\alpha_0\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t}\,\hat{a}^\dagger\Bigg)\left|\alpha_0\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t}\right\rangle \\
&= \Bigg(\frac{\hbar\,\Omega}{2}-\hbar\,\left|\alpha_0\right|^2\,(\Omega-\omega_\mathrm{L}) \\
&\qquad +\hbar\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t}\,\hat{a}^\dagger\,(\alpha_0\,\Omega-\alpha_0\,(\Omega-\omega_\mathrm{L})-\omega_\mathrm{L}\,\alpha_0)\Bigg)\left|\alpha_0\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t}\right\rangle \\
&= \hbar\left(\frac{\Omega}{2}-\left|\alpha_0\right|^2\,(\Omega-\omega_\mathrm{L})\right)\left|\alpha_0\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t}\right\rangle\;,
\end{aligned}$$

where we have used $L=-\alpha_0\,(\Omega-\omega_\mathrm{L})$. □

Proof of (34). The time-derivative of $\frac{\mathrm{d}}{\mathrm{d}t}\hat{b}_q(t)$ is now

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{b}_q(t)\overset{(37)}{=}-\mathrm{i}\,\omega_q\,\hat{b}_q(t)-\mathrm{i}\,\kappa_q^*\,\hat{a}(t)-\mathrm{i}\,\kappa_q^*\,\bar{a}_\mathrm{L}\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t}\;,$$

with solution

$$\hat{b}_q(t) = -\,\mathrm{i}\,\kappa_q^*\,\mathrm{e}^{-\mathrm{i}\,\omega_q\,t}\int^t \mathrm{e}^{\mathrm{i}\,\omega_q\,t'}\left(\hat{a}(t') + \bar{a}_\mathrm{L}\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t'}\right)\mathrm{d}t'$$

$$= -\,\mathrm{i}\,\kappa_q^*\int_{t_0}^t \mathrm{e}^{-\mathrm{i}\,\omega_q\,(t-t')}\,\hat{a}(t')\,\mathrm{d}t' - \mathrm{i}\,\kappa_q^*\,\bar{a}_\mathrm{L}\int_{t_0}^t \mathrm{e}^{-\mathrm{i}\,\omega_q\,(t-t')-\mathrm{i}\,\omega_\mathrm{L}\,t'}\,\mathrm{d}t' + \mathrm{e}^{-\mathrm{i}\,\omega_q\,(t-t_0)}\,\hat{b}_q(t_0)$$

$$= -\,\mathrm{i}\,\kappa_q^*\int_{t_0}^t \mathrm{e}^{-\mathrm{i}\,\omega_q\,(t-t')}\,\hat{a}(t')\,\mathrm{d}t' - \frac{\kappa_q^*\,\bar{a}_\mathrm{L}\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t}}{\omega_q - \omega_\mathrm{L}}\left(1 - \mathrm{e}^{-\mathrm{i}\,(t-t_0)\,(\omega_q-\omega_\mathrm{L})}\right) + \mathrm{e}^{-\mathrm{i}\,\omega_q\,(t-t_0)}\,\hat{b}_q(t_0)\,.$$

The time-derivative for $\hat{a}(t)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}(t) = -\,\mathrm{i}\,\Omega\,\hat{a}(t) - \mathrm{i}\,(\Omega-\omega_\mathrm{L})\,\bar{a}_\mathrm{L}\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t} - \mathrm{i}\,L\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t} - \sum_q \mathrm{i}\,\kappa_q\,\hat{b}_q(t)\,.$$

Applying the continuum and Markovian approximations to the last term,

$$-\sum_q \mathrm{i}\,\kappa_q\,\hat{b}_q(t) \approx -\,|\kappa|^2\int_{t_0}^t\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\,\omega\,(t-t')}\,\mathrm{d}\omega\,\hat{a}(t')\,\mathrm{d}t' + \mathrm{i}\;|\kappa|^2\;\bar{a}_\mathrm{L}\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t}\int_{-\infty}^{\infty}\frac{1-\mathrm{e}^{-\mathrm{i}\,(t-t_0)\,(\omega-\omega_\mathrm{L})}}{\omega-\omega_\mathrm{L}}\,\mathrm{d}\omega - \mathrm{i}\,\kappa\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\,\omega\,(t-t_0)}\,\hat{b}(\omega,t_0)\,\mathrm{d}\omega$$

$$= -\,\pi\;|\kappa|^2\;\hat{a}(t) + \mathrm{i}\;|\kappa|^2\;\bar{a}_\mathrm{L}\,\mathrm{e}^{-\mathrm{i}\,\omega_\mathrm{L}\,t}\int_{-\infty}^{\infty}\frac{1-\mathrm{e}^{-\mathrm{i}\,(t-t_0)\,(\omega-\omega_\mathrm{L})}}{\omega-\omega_\mathrm{L}}\,\mathrm{d}\omega - \sqrt{2\,\pi}\,\mathrm{i}\,\kappa\,\hat{b}_\mathrm{in}(t)\,.$$

The integral can be evaluated as follows,

$$\int_{-\infty}^{\infty}\frac{1-\mathrm{e}^{-\mathrm{i}\,(t-t_0)\,(\omega-\omega_\mathrm{L})}}{\omega-\omega_\mathrm{L}}\,\mathrm{d}\omega = \lim_{x\to\infty}\int_{-x}^{x}\frac{1-\mathrm{e}^{\mathrm{i}\,\theta}}{\theta}\,\mathrm{d}\theta \quad : \quad \theta = -(t-t_0)\,(\omega-\omega_\mathrm{L})$$

$$= \lim_{x\to\infty}\int_{-x}^{x}\frac{1-\cos[\theta]}{\theta}\,\mathrm{d}\theta + \lim_{x\to\infty}\mathrm{i}\int_{-x}^{x}\frac{\sin[\theta]}{\theta}\,\mathrm{d}\theta$$

and

$$\frac{1-\cos[(-\theta)]}{(-\theta)} = -\frac{1-\cos[\theta]}{\theta} \quad\Rightarrow\quad \int_{-x}^{x}\frac{1-\cos[\theta]}{\theta}\,\mathrm{d}\theta = 0\,,$$

$$\frac{\sin[(-\theta)]}{(-\theta)} = \frac{\sin[\theta]}{\theta} \quad\Rightarrow\quad \int_{-x}^{x}\frac{\sin[\theta]}{\theta}\,\mathrm{d}\theta = 2\int_{0}^{x}\frac{\sin[\theta]}{\theta}\,\mathrm{d}\theta\,.$$

This non-zero integral is the Sine Integral, which has known limit,

$$\int_0^x \frac{\sin[\theta]}{\theta}\,\mathrm{d}\theta = \mathrm{Si}[x]\,, \qquad\qquad \lim_{x\to\infty} \mathrm{Si}[x] = \frac{\pi}{2}\,.$$

Putting this all together, we find

$$\int_{-\infty}^{\infty} \frac{1-\mathrm{e}^{-\mathrm{i}\,(t-t_0)\,(\omega-\omega_j)}}{\omega-\omega_j}\,\mathrm{d}\omega = \mathrm{i}\,\pi\,,$$

and thus

$$-\sum_q \mathrm{i}\,\kappa_q\,\hat{b}_q(t) \approx -\pi\,|\kappa|^2\,\hat{a}(t) - \pi\,|\kappa|^2\,\bar{a}_{\mathrm{L}}\,\mathrm{e}^{-\mathrm{i}\,\omega_{\mathrm{L}}\,t} - \sqrt{2\,\pi}\,\mathrm{i}\,\kappa\,\hat{b}_{\mathrm{in}}(t)\,.$$

Putting this into the time-derivative for $\hat{a}(t)$, with $\gamma = -2\,\pi\,\kappa^2$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}(t) = &-\mathrm{i}\,\Omega\,\hat{a}(t) - \frac{|\gamma|}{2}\,\hat{a}(t) - \sqrt{\gamma}\,\hat{b}_{\mathrm{in}}(t)\\ &\underbrace{-\left(\mathrm{i}\,(\Omega-\omega_{\mathrm{L}}) + \frac{|\gamma|}{2}\right)\bar{a}_{\mathrm{L}}\,\mathrm{e}^{-\mathrm{i}\,\omega_{\mathrm{L}}\,t} - \mathrm{i}\,L\,\mathrm{e}^{-\mathrm{i}\,\omega_{\mathrm{L}}\,t}}_{0}\,,\end{aligned}$$

where the last line is zero by

$$\bar{a}_{\mathrm{L}} = \frac{-\mathrm{i}\,L}{\mathrm{i}\,(\Omega-\omega_{\mathrm{L}}) + \frac{|\gamma|}{2}}\,.$$

□

Proof of (35).

$$\begin{aligned}\left[\frac{\partial}{\partial t}, \hat{A}(t)\right] f &= \frac{\partial}{\partial t}\hat{A}(t)\,f - \hat{A}(t)\,\frac{\partial}{\partial t} f\\ &= \left(\frac{\partial}{\partial t}\hat{A}(t)\right) f + \hat{A}(t)\,\frac{\partial}{\partial t} f - \hat{A}(t)\,\frac{\partial}{\partial t} f\\ &= \left(\frac{\partial}{\partial t}\hat{A}(t)\right) f\,.\end{aligned}$$

□

Proof of (38).

$$\begin{aligned}0 &= \left(\hat{H}(t) - \mathrm{i}\,\hbar\,\frac{\mathrm{d}}{\mathrm{d}t}\right)|\psi(t)\rangle\\ &= \left(\hat{H}(t) - \mathrm{i}\,\hbar\,\frac{\mathrm{d}}{\mathrm{d}t}\right)\hat{U}(t)\,|\psi'(t)\rangle\\ &= \hat{H}(t)\,\hat{U}(t)\,|\psi'(t)\rangle - \left(\mathrm{i}\,\hbar\,\frac{\mathrm{d}}{\mathrm{d}t}\hat{U}(t)\right)|\psi'(t)\rangle - \hat{U}(t)\,\mathrm{i}\,\hbar\,\frac{\mathrm{d}}{\mathrm{d}t}\,|\psi'(t)\rangle\\ &= \hat{U}^\dagger(t)\,\hat{H}(t)\,\hat{U}(t)\,|\psi'(t)\rangle - \hat{U}^\dagger(t)\left(\mathrm{i}\,\hbar\,\frac{\mathrm{d}}{\mathrm{d}t}\hat{U}(t)\right)|\psi'(t)\rangle - \mathrm{i}\,\hbar\,\frac{\mathrm{d}}{\mathrm{d}t}\,|\psi'(t)\rangle\\ &= \left(\hat{H}'(t) - \mathrm{i}\,\hbar\,\frac{\mathrm{d}}{\mathrm{d}t}\right)|\psi'(t)\rangle\,.\end{aligned}$$

□

Proof of (39). We want to illuminate the meaning of

$$\langle \hat{\partial} f \,|\, g \rangle = \langle f \,|\, \hat{\partial}^\dagger g \rangle \ .$$

Using integration by parts

$$\begin{aligned}
\langle \hat{\partial} f \,|\, g \rangle &= \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial x} f(x) \right)^* g(x)\, \mathrm{d}x \\
&= \underbrace{f(x)^*\, g(x) \Big|_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty} f(x)^* \left(\frac{\partial}{\partial x} g(x) \right) \mathrm{d}x \\
&= \langle f \,|\, -\hat{\partial} g \rangle \ ,
\end{aligned}$$

when f and g are appropriately normalizable functions in quantum mechanics with $f(x \to \pm\infty) \to 0$ and $g(x \to \pm\infty) \to 0$. This allows us to write things like

$$\left(\frac{\partial}{\partial x} \right)^\dagger = -\frac{\partial}{\partial x} \ , \qquad\qquad \left(\frac{\partial}{\partial x} f(x) \right)^\dagger = -f(x)^\dagger \frac{\partial}{\partial x} \ .$$

□

Proof of (43). To determine $\Theta[0]$, consider

$$\int_{-\infty}^{\infty} \delta[x]\, \Theta[x]\, \mathrm{d}x = \Theta[0] \ .$$

By integration by parts, this is

$$\int_{-\infty}^{\infty} \Theta[x] \left(\delta[x]\, \mathrm{d}x \right) = \Theta[x]\, \Theta[x] \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta[x]\, \Theta[x]\, \mathrm{d}x \ ,$$

$$\Rightarrow \quad 2 \int_{-\infty}^{\infty} \delta[x]\, \Theta[x]\, \mathrm{d}x = 1 \ ,$$

thus

$$\Theta[0] = \frac{1}{2} \ .$$

□

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